Vertex-disjoint path covers in graphs*

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Abstract. *Let G be a connected graph and $\mathcal{P}$ be a set of pairwise vertex-disjoint paths in G. We say that $\mathcal{P}$ is a* path cover *if every vertex of G belongs to a path in $\mathcal{P}$. The* minimum path cover problem *asks for a path cover of minimum cardinality. In this problem, known to be NP-hard, the set $\mathcal{P}$ may contain trivial (single-vertex) paths. We study a variant of this problem in which the objective is to find a path cover without trivial paths. Using the well-known Edmonds-Gallai decomposition, we show that deciding whether a graph contains such kind of path cover can be reduced to a matching problem on a bipartite graph. We also show hardness and inapproximability results for both problems.*

Resumo. *Seja G um grafo conexo e $\mathcal{P}$ um conjunto de caminhos disjuntos nos vértices em G. Dizemos que $\mathcal{P}$ é uma* cobertura por caminhos *se cada vértice de G pertence a algum caminho em $\mathcal{P}$. No problema da* cobertura mínima por caminhos*, o objetivo é encontrar uma cobertura com o menor número de caminhos. Nesse problema, que é sabido ser NP-difícil, o conjunto $\mathcal{P}$ pode conter caminhos triviais. Estudamos uma variante desse problema onde o objetivo é encontrar uma cobertura sem caminhos triviais. Usando a decomposição de Edmonds-Gallai, mostramos que o problema de decidir se um grafo tem tal cobertura pode ser reduzido a um problema de emparelhamento em um grafo bipartido. Além disso, mostramos resultados de inaproximabilidade para ambos os problemas de cobertura: com e sem caminhos triviais.*

1. Introduction

All graphs considered here are simple and undirected. Let G be a graph and P be a path in G. The *length* of P is its number of edges. If P has length k, we say that it is a k-*path*. In the special case when $k = 0$, we also say that P is *trivial*. A set of pairwise vertex-disjoint paths is a *path cover* of G if it spans $V(G)$.

In the MINIMUM PATH COVER (MINPC) problem we want to find a path cover of minimum cardinality. Clearly, deciding whether a graph has a Hamiltonian path is equivalent to deciding whether it has a path cover of cardinality one. Therefore, MINPC is NP-hard in the classes of graphs for which the Hamiltonian path problem is NP-complete, such as cubic planar 3-connected graphs [Garey et al. 1976], circle graphs [Damaschke 1989], split graphs, chordal bipartite graphs [Müller 1996], etc.

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Polynomial-time algorithms have been designed for MINPC on several classes of perfect graphs, such as interval graphs [Arikati and Pandu Rangan 1990], cocomparability graphs [Corneil et al. 2013], trees [Franzblau and Raychaudhuri 2002], etc. This problem is of theoretical, but also of practical interest. It has applications in establishing ring protocols in a network, code optimization and mapping parallel processes to parallel architectures [Moran and Wolfstahl 1991].

Besides studying MINPC, we also study the problem of finding a path cover without trivial paths. First, we consider the problem of the existence of such a path cover. Then, we consider the two optimization problems: the MINIMUM NONTRIVIAL PATH COVER (MINNTPC) and the MAXIMUM NONTRIVIAL PATH COVER (MAXNTPC). In both cases we seek for path covers of optimum cardinality. In the next section, we show that the existence problem and MAXNTPC have a close relation with the maximum matchings of a graph.

2. Covering a graph with nontrivial paths

Let g and f be integer functions defined on the vertices of a graph G. A (g, f)-*factor* of G is a subgraph H of G such that $V(H) = V(G)$ and each vertex x has degree at least $g(x)$ and at most $f(x)$ in H. Let a and b be two integers such that $a \leq b$. When $g(x) = a$ and $f(x) = b$ for every x in G, we say that H is an $[a, b]$-*factor*. Thus, deciding whether a graph has a nontrivial path cover is equivalent to deciding whether it has a $[1, 2]$-factor.

There is a large number of results characterizing graphs which contain (g, f)-factors: both algorithmic and structural ones. In 1952, Tutte characterized graphs that have an (f, f)-factor. Later, [Lovász 1970] gave a characterization of the graphs which admit a (g, f)-factor. In the eighties, [Anstee 1985] showed an algorithm that finds a (g, f)-factor, if it exists, or finds a certificate of its non-existence in polynomial time. Another algorithmic result was given by [Heinrich et al. 1990], who designed a more efficient algorithm for the special case in which $g(x) \leq 1$ and $g(x) < f(x)$ for every vertex x in the graph. Therefore, it has already been proved that the nontrivial path cover existence problem can be solved in polynomial time.

We show that the existence of a nontrivial path cover in a graph has a close relation with the structure of its maximum matchings. Let G be a graph that has a path cover by k-paths with $k \geq 1$. Note that, the paths of length greater than 2 can be broken into paths of length 1 or 2. Therefore, we may focus only on the problem of deciding whether there exists a path cover of G consisting only of 1-paths or 2-paths. Let us denote by $\overline{\mathcal{P}}_{1,2}(G)$ a path cover of G composed by 1-paths or 2-paths with the property that it has the largest possible number of 1-paths. We can prove the following result on $|\overline{\mathcal{P}}_{1,2}(G)|$.

Proposition 1 *Let G be a graph that admits a nontrivial path cover, and let $\overline{\mathcal{P}}_{1,2}(G)$ be a path cover as defined above. Then $|\overline{\mathcal{P}}_{1,2}(G)|$ is precisely the cardinality of a maximum matching of G.*

By Proposition 1, if we choose an edge from every path in $\overline{\mathcal{P}}_{1,2}(G)$, we obtain a maximum matching of G. Therefore, if a graph G admits a nontrivial path cover, then G has a maximum matching that can be extended to a path cover $\overline{\mathcal{P}}_{1,2}(G)$. However, it is not true that every maximum matching has this property. Using the structure given by the Edmonds-Gallai decomposition [Lovász and Plummer 1986] of a graph, we can reduce

MAXNTPC on G to a maximum matching problem in a bipartite graph, and prove the following result.

Theorem 1 *The* MAXNTPC *problem can be solved in polynomial time.*

The classic result of [Lovász 1970] characterizing graphs that admit a (g, f)-factor gives the following result when we consider the special case of $[1, 2]$-factors (see [Las Vergnas 1978]).

Theorem 2 (Lovász 1970) *A graph G has a $[1, 2]$-factor, if and only if, $i(G - S) \leq 2|S|$ for every $S \subseteq V(G)$, where $i(G - S)$ is the number of isolated vertices in $G - S$.*

Our approach of specializing to path covers $\overline{\mathcal{P}}_{1,2}(G)$ of a graph G gives an efficient way to find a $[1, 2]$-factor (nontrivial path cover), when it exists, or to find a negative certificate. The next theorem shows how this certificate can be obtained.

Theorem 3 *Let G be a graph, $D(G)$ be the set given by the Edmonds-Gallai decomposition of G, and $T \subseteq D(G)$ be the set of vertices corresponding to the trivial hypomatchable components in $D(G)$. Then the following holds:*

(i) *G has a $[1, 2]$-factor if and only if $|X| \leq 2|N(X)|$, for every $X \subseteq T$.*
(ii) *If G does not have a $[1, 2]$-factor, and X is a set that violates the condition stated in* (i)*, then $S = N(X)$ (the set of neighbours of X) is a set that violates the condition stated in Theorem 2. Moreover, S can be found in polynomial time.*

Next, we show a reduction from MINPC to MINNTPC. Let G be a cubic graph. Let $G' = (V', E')$ be a cubic graph obtained from G by replacing each vertex v in $V(G)$ with a triangle T_v (that is, a K_3). Each vertex of T_v represents one of the edges incident to v in G. The set of edges E' is composed of the edges of the triangles T_v for each v in $V(G)$, and for each edge $uv \in E(G)$ there is an edge in E' linking the vertices in T_u and T_v that represent the edge uv. Observe that G' always has a nontrivial path cover. Moreover, if G is planar and 3-connected, then G' also has these properties. The following result is used to show the hardness to approximate MINNTPC.

Proposition 2 *Let G be a cubic graph and let G' be a graph as defined above. Let $\mathcal{P}$ be a nontrivial path cover of G'. Then, G' has a nontrivial path cover $\mathcal{Q}$ such that, $|\mathcal{Q}| \leq |\mathcal{P}|$, and whenever a path in $\mathcal{Q}$ intersects a triangle T_v, it traverses two edges of T_v.*

Using the reduction and the Proposition above, we can conclude that MINNTPC is NP-hard on cubic planar 3-connected graphs. This result also follows from the fact that the Hamiltonian path problem is NP-complete on cubic planar 3-connected graphs.

Now, we present some inapproximability results for MINPC and MINNTPC. For that, we use the fact that the MINIMUM TRAVELING SALESMAN (MINTSP-$(1, 2)$) problem is Max SNP-hard [Papadimitriou and Yannakakis 1993] even when the edge-weights are 1 or 2, and the edges with weight 1 induce a cubic graph. Using a result of Arora et al. [Arora et al. 1998] and showing AP-reductions from MINTSP-$(1, 2)$ to MINPC, and from MINPC to MINNTPC, we prove the following result.

Theorem 4 MINPC *and* MINNTPC *on cubic graphs have no* PTAS*, unless* P $=$ NP.

We also show an approximation threshold for MINPC.

Theorem 5 *For every $\epsilon > 0$, there is no $\left(\frac{371}{370} - \epsilon\right)$-approximation algorithm for* MINPC*, unless* P = NP.

When restricted to trees, we can show the following result.

Theorem 6 MINNTPC *on trees can be solved in linear time.*

3. Concluding remarks

As far as we know, the MINNTPC and MAXNTPC problems have not been treated in the literature. Currently, we are testing some integer programming formulations which we have proposed for MINNTPC. The computational results are preliminary, but seem very promising. We are also interested in the design of approximation algorithms for MINPC and MINNTPC.

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