# On Total and Edge-colouring of Proper Circular-arc Graphs 

João Pedro W. Bernardi ${ }^{1,2 *}$, Sheila M. de Almeida ${ }^{3 *}$, Leandro M. Zatesko ${ }^{1,2 * \dagger}$<br>${ }^{1}$ Federal University of Fronteira Sul, Chapecó, Brazil<br>${ }^{2}$ Federal University of Paraná, Curitiba, Brazil<br>${ }^{3}$ Federal University of Technology — Paraná, Ponta Grossa, Brazil

\{winckler,leandro.zatesko\}@ufpr.br, sheilaalmeida@utfpr.edu.br


#### Abstract

Deciding if a graph is $\Delta$-edge-colourable (resp. $(\Delta+1)$-total colourable), although it is an NP-complete problem for graphs in general, is polynomially solvable for interval graphs of odd (resp. even) maximum degree $\Delta$. An interesting superclass of the proper interval graphs are the proper circular-arc graphs, for which we suspect that $\Delta$-edge-colourability is linear-time decidable. This work presents sufficient conditions for $\Delta$-edge-colourability, $(\Delta+1)$-total colourability, and ( $\Delta+2$ )-total colourability of proper circular-arc graphs. Our proofs are constructive and yield polynomial-time algorithms.


## 1. Introduction

The chromatic index and the total chromatic number of a graph $G$ with maximum degree $\Delta$ clearly satisfy $\chi^{\prime}(G) \geq \Delta$ and $\chi^{\prime \prime}(G) \geq \Delta+1$ (see definitions in the sequel). Also, $\chi^{\prime}(G) \leq \Delta+1$ [Vizing 1964], and the Total Colouring Conjecture states that $\chi^{\prime \prime}(G) \leq$ $\Delta+2$ [Behzad 1965, Vizing 1968]. A graph $G$ is Class 1 if $\chi^{\prime}(G)=\Delta$, or Class 2 otherwise. Since no graph with $\chi^{\prime \prime}(G) \geq \Delta+3$ is known, graphs with $\chi^{\prime \prime}(G)=\Delta+1$ have been called Type 1, and those with $\chi^{\prime \prime}(G)=\Delta+2$ Type 2. Deciding if $G$ is Class 1 and deciding if $G$ is Type 1 are NP-complete problems [Holyer 1981, Sánchez-Arroyo 1989].

The classes of the unit and the proper interval graphs are the same [Roberts 1969], but the classes of the unit and the proper circular-arc graphs are not (see Figure 1). The


Figure 1. A proper non-unit circular-arc graph with a corresponding arc model
Total Colouring Conjecture holds for proper interval graphs, often referred to as indifference graphs, which are Class 1 when they have odd $\Delta$, and Type 1 when $\Delta$ is even [Figueiredo et al. 1997]. For edge-colouring of indifference graphs with even $\Delta$ or total colouring of these graphs with odd $\Delta$, partial results are known [Figueiredo et al. 2003,

[^0]Campos et al. 2012]. Recall that interval graphs are perfect graphs and, thus, admit poly-nomial-time vertex-colouring algorithms [Grötschel et al. 1981], in contrast to vertex-colouring of circular-arc graphs, which is NP-hard [Garey et al. 1980]. To the best of our knowledge, there is no published work on total or edge-colouring of circular-arc graphs.

Let $G$ be an $n$-vertex proper circular-arc graph. We show that if $n \equiv 0(\bmod (\Delta+$ $1)$ ), or if $G$ has a maximal clique of size 2 and $n \not \equiv k(\bmod (\Delta+1))$ for all $k \in\{1, \Delta\}$, then: $\chi^{\prime}(G)=\Delta$ and $\chi^{\prime \prime}(G) \leq \Delta+2$ if $\Delta$ is odd; $\chi^{\prime \prime}(G)=\Delta+1$ if $\Delta$ is even. This implies that the Total Colouring Conjecture holds for the class of all such graphs.

This paper is organised as follows: the remaining of this section provides further definitions and discusses other related results in the literature; Section 2 presents our results; at last, Section 3 makes remarks on edge-colouring proper circular-arc graphs.

## Preliminary definitions and other related results

This work deals only with simple graphs, referred to simply as graphs. Usual terms concerning graph-theoretical concepts follow their definitions and notation in the literature. In particular, the degree of a vertex $u$ in a graph $G$, the set of neighbours of $u$ in $G$, and the set of the edges incident to $u$ in $G$ are denoted by $d_{G}(u), N_{G}(u)$, and $\partial_{G}(u)$, respectively.

Let $G=(V, E)$ be a graph and $\mathscr{C}$ a set of $t$ colours. A function with $\mathscr{C}$ as its codomain is: a t-edge-colouring if its domain is $E$ and it is injective in $\partial_{G}(u)$ for all $u \in V$; a $t$-total colouring if its domain is $V \cup E$ and it is injective in $\partial_{G}(u) \cup\{u\}$ for all $u \in V$ and injective in $\{u, v\}$ for all $u v \in E$. In a total or edge-colouring, we say that a colour is missing at a vertex $u$ if it is not assigned to $u$ or to any edge incident to $u$. The chromatic index (denoted by $\chi^{\prime}(G)$ ) and the total chromatic number (denoted by $\chi^{\prime \prime}(G)$ ) of $G$ are the least $t$ for which $G$ is $t$-edge-colourable and $t$-total colourable, respectively.

An $n$-vertex graph with more than $\Delta\lfloor n / 2\rfloor$ edges (thus Class 2 , since at most $\lfloor n / 2\rfloor$ edges can be coloured the same) is said to be overfull. It is conjectured that every graph $G$ with $\Delta>n / 3$ is Class 2 if and only if it is subgraph-overfull (shortly, SO), i.e. if $G$ has an overfull subgraph with the same maximum degree [Hilton and Johnson 1987].

The complete graph $K_{n}$ is: Class 1 and Type 2 if $n$ is even; Class 2 and Type 1 if $n$ is odd [Behzad et al. 1967]. Let $V\left(K_{n}\right):=\{0, \ldots, n-1\}$ and let even $(n)$ be 1 if $n$ is even or 0 otherwise. We call the canonical total and edge-colourings of the $K_{n}$ the functions $\varphi_{\text {edge }}^{\text {even }}, \varphi_{\text {edge }}^{\text {odd }}$, and $\varphi_{\text {total }}$ given by: $\varphi_{\text {edge }}^{\text {even }}(u v):=(u+v) \bmod (n-1)$, if neither $u$ nor $v$ is $n-1 ; \varphi_{\text {edge }}^{\text {even }}(u v):=(2 u) \bmod (n-1)$, if $v=n-1 ; \varphi_{\text {edge }}^{\text {odd }}(u v)=\varphi_{\text {total }}(u v):=$ $(u+v) \bmod (n+\operatorname{even}(n)) ; \varphi_{\text {total }}(u):=(2 u) \bmod (n+\operatorname{even}(n))$.

A circular-arc graph $G$ is the intersection graph of a finite set $S$ of arcs of a circle, in which case $S$ is an arc model corresponding to $G$. Furthermore, $G$ is said to be: proper, if there is a corresponding arc model wherein no arc properly contains another; a unit circular-arc graph, if there is a model wherein all the arcs have equal length. The vertices of a proper circular-arc graph admit a proper circular-arc order, i.e. a circular order in which vertices belonging to the same maximal clique appear consecutively. Homonymous terms are defined for interval graphs analogously, but being $S$ a set of intervals on the real line and the proper interval (or indifference) order a linear order. Interval and circular-arc graphs can be recognised in linear time [Booth and Lueker 1976, McConnell 2003].

A pullback from $G_{1}=\left(V_{1}, E_{1}\right)$ to $G_{2}=\left(V_{2}, E_{2}\right)$ is a homeomorphism $\pi$ : $V_{1} \rightarrow$
$V_{2}$ (i.e. $\pi(u) \pi(v) \in E_{2}$ for all $u v \in E_{1}$ ) injective in $N_{G_{1}}(u) \cup\{u\}$ for all $u \in V_{1}$. If such a pullback exists and $G_{2}$ has a: $t$-edge-colouring $\varphi$, then a $t$-edge-colouring for $G_{1}$ can be given by $\psi(u v):=\varphi(\pi(u) \pi(v)) ; t$-total colouring $\varphi$, then a $t$-total colouring for $G_{1}$ can be given by $\psi(u v):=\varphi(\pi(u) \pi(v))$ and $\psi(u):=\varphi(\pi(u))$ [Figueiredo et al. 1997].

## 2. Results

Throughout this section, let $G$ be an $n$-vertex proper circular-arc graph. Remark that when we say that $G$ is $\Delta+2$-total colourable, it does not mean that $G$ is Type 2 .

Theorem 1. If $n \equiv 0(\bmod (\Delta+1))$, then $G$ is: Class 1 and $(\Delta+2)$-total colourable if $\Delta$ is odd; Type 1 if $\Delta$ is even.
Proof. It suffices to show that if $n \equiv 0(\bmod (\Delta+1))$, then there is a pullback from $G$ to the $K_{\Delta+1}$. Let $\sigma:=u_{0}, \ldots, u_{n-1}$ be a proper circular-arc order of $G$ and $0, \ldots, \Delta$ be the vertices of the $K_{\Delta+1}$. Assume, by the sake of contradiction, that the function $\pi: V(G) \rightarrow V\left(K_{\Delta+1}\right)$ defined by $\pi\left(u_{i}\right):=i \bmod (\Delta+1)$ is not a pullback from $G$ to the $K_{\Delta+1}$. As $\pi$ is clearly a homeomorphism, there must be two distinct vertices $v_{1}$ and $v_{2}$ in $V(G)$ which have a neighbour $w$ in common and satisfy $\pi\left(v_{1}\right)=\pi\left(v_{2}\right)$. However, since $\sigma$ is a proper circular-arc order of $G$, all vertices between $v_{1}$ and $v_{2}$ in $\sigma$ are thus neighbours of $w$, which straightforwardly implies $d_{G}(w)>\Delta$.

Theorem 2. If $n \not \equiv k(\bmod (\Delta+1))$, for all $k \in\{1, \Delta\}$, and $G$ has a maximal clique of size 2 , then $G$ is: Class 1 and $(\Delta+2)$-total colourable if $\Delta$ is odd; Type 1 if $\Delta$ is even.

Proof. If $r:=n \bmod (\Delta+1)=0$, we are done by Theorem 1. If $\Delta \leq 2$, then $G$ is a cycle or a disjoint union of paths and the theorem clearly holds. Hence, we assume that $\Delta \geq 3$ and $r \neq 0$. Let $\sigma:=u_{0}, \ldots, u_{n-1}$ be a proper circular-arc order of $G$, being $\left\{u_{0}, u_{n-1}\right\}$ a maximal clique. Because $\sigma$ is a proper circular-arc order, we have $u_{\Delta} \notin N_{G}\left(u_{0}\right)$ and $u_{n-1-\Delta} \notin N_{G}\left(u_{n-1}\right)$, otherwise $d_{G}\left(u_{0}\right)>\Delta$ or $d_{G}\left(u_{n-1}\right)>\Delta$.

Let $V\left(K_{\Delta+1}\right):=\{0, \ldots, \Delta\}$ and let $\varphi \in\left\{\varphi_{\text {edge }}^{\text {even }}, \varphi_{\text {total }}\right\}$ be the canonical total or edge-colouring of the $K_{\Delta+1}$. The function $\pi: V\left(G^{\prime}\right) \rightarrow V\left(K_{\Delta+1}\right)$ defined by $\pi\left(u_{i}\right):=$ $i \bmod (\Delta+1)$ is clearly a pullback from $G^{\prime}:=G-u_{n-1} u_{0}$ to the $K_{\Delta+1}$ and brings a total or an edge-colouring $\psi$ of $G^{\prime}$ using the same set of colours as $\varphi$. Ergo, we have only to colour $u_{n-1} u_{0}$ in order to complete the proof.

Observe that $\pi\left(u_{n-1}\right)=r-1, \pi\left(u_{n-1-\Delta}\right)=r$, and, since neither $r$ nor $r-1$ is $\Delta$, $\varphi(r, r-1)=(2 r-1) \bmod d=: q$, with $d:=\Delta$ if $\varphi=\varphi_{\text {edge }}^{\text {even }}$, or $d:=\Delta+1+\operatorname{even}(\Delta+1)$ if $\varphi=\varphi_{\text {total }}$. Therefore, as $\pi(v) \neq \Delta$ and $\pi(w) \neq r$ for all $v \in N_{G^{\prime}}\left(u_{0}\right)$ and all $w \in$ $N_{G^{\prime}}\left(u_{n-1}\right)$, the colour $\varphi(0, \Delta)$ is missing at $u_{0}$ and the colour $q$ at $u_{n-1}$. If $q=\varphi(0, \Delta)$, then we assign the colour $q$ to $u_{n-1} u_{0}$ and we are done. Otherwise, since $q \in\{0, \ldots, \Delta\}$, we exchange $\Delta$ and $q$ in the codomain of $\pi$, that is, we redefine $\pi$ so that every vertex which has been mapped by $\pi$ to $\Delta$ is now mapped to $q$ and vice versa. Notice that the images of $u_{0}, u_{n-1-\Delta}$, and $u_{n-1}$ by $\pi$ remain the same, but $\pi\left(u_{\Delta}\right)$ becomes $q$, which now is also a colour missing at $u_{0}$. Then, we colour $u_{n-1} u_{0}$ with $q$.

## 3. Final remarks

Let $\mathcal{A}$ be the class of the proper circular-arc graphs with odd $\Delta$ and a maximal clique of size 2 . Overfull graphs in $\mathcal{A}$ can be constructed for $n \equiv 1$ and for $n \equiv \Delta(\bmod (\Delta+1))$ (see Figures 2(a) and 2(b), respectively). Since Theorem 2 can be interestingly used to
a)

b)


Figure 2. Two overfull graphs in $\mathcal{A}$
show a graph in $\mathcal{A}$ is $S O$ if and only if it is overfull, we conclude proposing the following:
Conjecture. A graph in $\mathcal{A}$ is Class 2 if and only if it is overfull.

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