On Total and Edge-colouring of Proper Circular-arc Graphs

João Pedro W. Bernardi^{1,2*}, Sheila M. de Almeida^{3*}, Leandro M. Zatesko^{1,2*†}

¹Federal University of Fronteira Sul, Chapecó, Brazil

²Federal University of Paraná, Curitiba, Brazil

³Federal University of Technology — Paraná, Ponta Grossa, Brazil

{winckler,leandro.zatesko}@ufpr.br, sheilaalmeida@utfpr.edu.br

Abstract. Deciding if a graph is Δ -edge-colourable (resp. $(\Delta+1)$ -total colourable), although it is an NP-complete problem for graphs in general, is polynomially solvable for interval graphs of odd (resp. even) maximum degree Δ . An interesting superclass of the proper interval graphs are the proper circular-arc graphs, for which we suspect that Δ -edge-colourability is linear-time decidable. This work presents sufficient conditions for Δ -edge-colourability, $(\Delta+1)$ -total colourability, and $(\Delta+2)$ -total colourability of proper circular-arc graphs. Our proofs are constructive and yield polynomial-time algorithms.

1. Introduction

The chromatic index and the total chromatic number of a graph G with maximum degree Δ clearly satisfy $\chi'(G) \geq \Delta$ and $\chi''(G) \geq \Delta + 1$ (see definitions in the sequel). Also, $\chi'(G) \leq \Delta + 1$ [Vizing 1964], and the *Total Colouring Conjecture* states that $\chi''(G) \leq \Delta + 2$ [Behzad 1965, Vizing 1968]. A graph G is Class 1 if $\chi'(G) = \Delta$, or Class 2 otherwise. Since no graph with $\chi''(G) \geq \Delta + 3$ is known, graphs with $\chi''(G) = \Delta + 1$ have been called *Type 1*, and those with $\chi''(G) = \Delta + 2$ *Type 2*. Deciding if G is Class 1 and deciding if G is *Type 1* are NP-complete problems [Holyer 1981, Sánchez-Arroyo 1989].

The classes of the unit and the proper interval graphs are the same [Roberts 1969], but the classes of the unit and the proper circular-arc graphs are not (see Figure 1). The



Figure 1. A proper non-unit circular-arc graph with a corresponding arc model

Total Colouring Conjecture holds for proper interval graphs, often referred to as *indif-ference graphs*, which are *Class 1* when they have odd Δ , and *Type 1* when Δ is even [Figueiredo et al. 1997]. For edge-colouring of indifference graphs with even Δ or total colouring of these graphs with odd Δ , partial results are known [Figueiredo et al. 2003,

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Campos et al. 2012]. Recall that interval graphs are perfect graphs and, thus, admit polynomial-time vertex-colouring algorithms [Grötschel et al. 1981], in contrast to vertex-colouring of circular-arc graphs, which is NP-hard [Garey et al. 1980]. To the best of our knowledge, there is no published work on total or edge-colouring of circular-arc graphs.

Let G be an n-vertex proper circular-arc graph. We show that if $n \equiv 0 \pmod{(\Delta + 1)}$, or if G has a maximal clique of size 2 and $n \not\equiv k \pmod{(\Delta + 1)}$ for all $k \in \{1, \Delta\}$, then: $\chi'(G) = \Delta$ and $\chi''(G) \leq \Delta + 2$ if Δ is odd; $\chi''(G) = \Delta + 1$ if Δ is even. This implies that the Total Colouring Conjecture holds for the class of all such graphs.

This paper is organised as follows: the remaining of this section provides further definitions and discusses other related results in the literature; Section 2 presents our results; at last, Section 3 makes remarks on edge-colouring proper circular-arc graphs.

Preliminary definitions and other related results

This work deals only with *simple graphs*, referred to simply as *graphs*. Usual terms concerning graph-theoretical concepts follow their definitions and notation in the literature. In particular, the *degree* of a vertex u in a graph G, the *set of neighbours* of u in G, and the *set of the edges incident* to u in G are denoted by $d_G(u)$, $N_G(u)$, and $\partial_G(u)$, respectively.

Let G = (V, E) be a graph and \mathscr{C} a set of t colours. A function with \mathscr{C} as its codomain is: a *t-edge-colouring* if its domain is E and it is injective in $\partial_G(u)$ for all $u \in V$; a *t-total colouring* if its domain is $V \cup E$ and it is injective in $\partial_G(u) \cup \{u\}$ for all $u \in V$ and injective in $\{u, v\}$ for all $uv \in E$. In a total or edge-colouring, we say that a colour is *missing* at a vertex u if it is not assigned to u or to any edge incident to u. The chromatic index (denoted by $\chi'(G)$) and the total chromatic number (denoted by $\chi''(G)$) of G are the least t for which G is *t*-edge-colourable and *t*-total colourable, respectively.

An *n*-vertex graph with more than $\Delta \lfloor n/2 \rfloor$ edges (thus *Class 2*, since at most $\lfloor n/2 \rfloor$ edges can be coloured the same) is said to be *overfull*. It is conjectured that every graph G with $\Delta > n/3$ is *Class 2* if and only if it is *subgraph-overfull* (shortly, *SO*), i.e. if G has an overfull subgraph with the same maximum degree [Hilton and Johnson 1987].

The complete graph K_n is: Class 1 and Type 2 if n is even; Class 2 and Type 1 if n is odd [Behzad et al. 1967]. Let $V(K_n) := \{0, \ldots, n-1\}$ and let even(n) be 1 if n is even or 0 otherwise. We call the canonical total and edge-colourings of the K_n the functions φ_{edge}^{even} , φ_{edge}^{odd} , and φ_{total} given by: $\varphi_{edge}^{even}(uv) := (u+v) \mod (n-1)$, if neither u nor v is n-1; $\varphi_{edge}^{even}(uv) := (2u) \mod (n-1)$, if v = n-1; $\varphi_{edge}^{odd}(uv) = \varphi_{total}(uv) := (u+v) \mod (n+even(n))$; $\varphi_{total}(u) := (2u) \mod (n+even(n))$.

A circular-arc graph G is the intersection graph of a finite set S of arcs of a circle, in which case S is an arc model corresponding to G. Furthermore, G is said to be: proper, if there is a corresponding arc model wherein no arc properly contains another; a unit circular-arc graph, if there is a model wherein all the arcs have equal length. The vertices of a proper circular-arc graph admit a proper circular-arc order, i.e. a circular order in which vertices belonging to the same maximal clique appear consecutively. Homonymous terms are defined for interval graphs analogously, but being S a set of intervals on the real line and the proper interval (or indifference) order a linear order. Interval and circular-arc graphs can be recognised in linear time [Booth and Lueker 1976, McConnell 2003].

A pullback from $G_1 = (V_1, E_1)$ to $G_2 = (V_2, E_2)$ is a homeomorphism $\pi \colon V_1 \to V_1$

 V_2 (i.e. $\pi(u)\pi(v) \in E_2$ for all $uv \in E_1$) injective in $N_{G_1}(u) \cup \{u\}$ for all $u \in V_1$. If such a pullback exists and G_2 has a: t-edge-colouring φ , then a t-edge-colouring for G_1 can be given by $\psi(uv) \coloneqq \varphi(\pi(u)\pi(v))$; t-total colouring φ , then a t-total colouring for G_1 can be given by $\psi(uv) \coloneqq \varphi(\pi(u)\pi(v))$ and $\psi(u) \coloneqq \varphi(\pi(u))$ [Figueiredo et al. 1997].

2. Results

Throughout this section, let G be an n-vertex proper circular-arc graph. Remark that when we say that G is $\Delta + 2$ -total colourable, it does not mean that G is Type 2.

Theorem 1. If $n \equiv 0 \pmod{(\Delta + 1)}$, then G is: Class 1 and $(\Delta + 2)$ -total colourable if Δ is odd; Type 1 if Δ is even.

Proof. It suffices to show that if $n \equiv 0 \pmod{(\Delta + 1)}$, then there is a pullback from G to the $K_{\Delta+1}$. Let $\sigma \coloneqq u_0, \ldots, u_{n-1}$ be a proper circular-arc order of G and $0, \ldots, \Delta$ be the vertices of the $K_{\Delta+1}$. Assume, by the sake of contradiction, that the function $\pi \colon V(G) \to V(K_{\Delta+1})$ defined by $\pi(u_i) \coloneqq i \mod (\Delta + 1)$ is *not* a pullback from G to the $K_{\Delta+1}$. As π is clearly a homeomorphism, there must be two distinct vertices v_1 and v_2 in V(G) which have a neighbour w in common and satisfy $\pi(v_1) = \pi(v_2)$. However, since σ is a proper circular-arc order of G, all vertices between v_1 and v_2 in σ are thus neighbours of w, which straightforwardly implies $d_G(w) > \Delta$.

Theorem 2. If $n \not\equiv k \pmod{(\Delta+1)}$, for all $k \in \{1, \Delta\}$, and G has a maximal clique of size 2, then G is: Class 1 and $(\Delta+2)$ -total colourable if Δ is odd; Type 1 if Δ is even.

Proof. If $r \coloneqq n \mod (\Delta + 1) = 0$, we are done by Theorem 1. If $\Delta \leq 2$, then G is a cycle or a disjoint union of paths and the theorem clearly holds. Hence, we assume that $\Delta \geq 3$ and $r \neq 0$. Let $\sigma \coloneqq u_0, \ldots, u_{n-1}$ be a proper circular-arc order of G, being $\{u_0, u_{n-1}\}$ a maximal clique. Because σ is a proper circular-arc order, we have $u_{\Delta} \notin N_G(u_0)$ and $u_{n-1-\Delta} \notin N_G(u_{n-1})$, otherwise $d_G(u_0) > \Delta$ or $d_G(u_{n-1}) > \Delta$.

Let $V(K_{\Delta+1}) \coloneqq \{0, \ldots, \Delta\}$ and let $\varphi \in \{\varphi_{\text{edge}}^{\text{even}}, \varphi_{\text{total}}\}$ be the canonical total or edge-colouring of the $K_{\Delta+1}$. The function $\pi \colon V(G') \to V(K_{\Delta+1})$ defined by $\pi(u_i) \coloneqq i \mod (\Delta + 1)$ is clearly a pullback from $G' \coloneqq G - u_{n-1}u_0$ to the $K_{\Delta+1}$ and brings a total or an edge-colouring ψ of G' using the same set of colours as φ . Ergo, we have only to colour $u_{n-1}u_0$ in order to complete the proof.

Observe that $\pi(u_{n-1}) = r-1$, $\pi(u_{n-1-\Delta}) = r$, and, since neither r nor r-1 is Δ , $\varphi(r, r-1) = (2r-1) \mod d =: q$, with $d := \Delta$ if $\varphi = \varphi_{\text{edge}}^{\text{even}}$, or $d := \Delta + 1 + \text{even}(\Delta + 1)$ if $\varphi = \varphi_{\text{total}}$. Therefore, as $\pi(v) \neq \Delta$ and $\pi(w) \neq r$ for all $v \in N_{G'}(u_0)$ and all $w \in N_{G'}(u_{n-1})$, the colour $\varphi(0, \Delta)$ is missing at u_0 and the colour q at u_{n-1} . If $q = \varphi(0, \Delta)$, then we assign the colour q to $u_{n-1}u_0$ and we are done. Otherwise, since $q \in \{0, \dots, \Delta\}$, we exchange Δ and q in the codomain of π , that is, we redefine π so that every vertex which has been mapped by π to Δ is now mapped to q and vice versa. Notice that the images of $u_0, u_{n-1-\Delta}$, and u_{n-1} by π remain the same, but $\pi(u_\Delta)$ becomes q, which now is also a colour missing at u_0 . Then, we colour $u_{n-1}u_0$ with q.

3. Final remarks

Let \mathcal{A} be the class of the proper circular-arc graphs with odd Δ and a maximal clique of size 2. Overfull graphs in \mathcal{A} can be constructed for $n \equiv 1$ and for $n \equiv \Delta \pmod{(\Delta + 1)}$ (see Figures 2(a) and 2(b), respectively). Since Theorem 2 can be interestingly used to



Figure 2. Two overfull graphs in \mathcal{A}

show a graph in A is SO if and only if it is overfull, we conclude proposing the following:

Conjecture. A graph in A is Class 2 if and only if it is overfull.

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