An infinite family of Type 2 snarks with small girth obtained by Kochol's Superposition

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Abstract. *Snarks are cubic graphs with peculiar properties, making them relevant to several problems in graph theory, such as edge and total coloring. While infinite families of Type 1 snarks are well known, Type 2 snarks remain rare and difficult to construct. In 1996, Kochol introduced a method for constructing new snarks by combining two known snarks, usually Type 1. We apply Kochol's superposition to obtain new Type 2 snarks. As a result, we construct an infinite family of Type 2 snarks with girth 4 by Kochol's superposition of a known Type 2 snark with the infinite family of Goldberg snarks.*

1. Introduction

Snarks are cubic, cyclically 4-edge-connected, Class 2 graphs, that do not admit a 3-coloring edges. For more details about them, see [Brinkmann et al. 2013]. For nearly 100 years, only five snarks had been identified. In [Isaacs 1975], the author introduced an operation that allowed the construction of new snarks, including the first infinite families. Later, in [Kochol 1996], a new method was proposed for generating snarks from smaller graphs, known as *Kochol's superposition*. However, the snarks constructed using this method, usually, are Type 1.

The goal is to adapt Kochol's superposition to construct Type 2 snarks. To this end, we apply Kochol's superposition to the known family of Goldberg snarks, along with the girth 4 Type 2 snark constructed by Brinkmann, Preissmann, and Sasaki [Brinkmann et al. 2015], which we refer to as *brick snark*.

2. Preliminaries

A *semi-graph* is a 3-tuple $G = (V, E, S)$, where V is the set of vertices of G, E is a set of edges disjoint from V, and S is a set of *semiedges*, each having exactly one endpoint in V. The *girth* of G is the length of the shortest cycle contained in G.

A k*-total coloring* is an assignment of k colors to the edges and vertices of a graph G, so that adjacent or incident elements have different colors. The *total chromatic number* of G, denoted by $\chi''(G)$, is the smallest value of k for which G admits a k-total coloring. Clearly, $\chi''(G) \geq \Delta(G) + 1$, where $\Delta(G)$ denotes the maximum degree of G. The upper bound was independently established by Vizing and Behzad through the famous *Total Coloring Conjecture [TCC]*, which states that $\chi''(G) \leq \Delta(G) + 2$ for any simple graph G. Although the conjecture remains open in general, it has been proved for cubic graphs, implying that their total chromatic number is either 4 (called *Type 1* graphs) or 5 (called *Type 2* graphs).

3. Kochol's superposition

Kochol's superposition consists of two main elements: a *superedge* ξ is a semi-graph with two connectors, and a *supervertex* ϑ is a semi-graph with three connectors. We consider the following semi-graphs depicted in Figure 1:

i. $(3, 3)$-semi-graph M' (superedge) is obtained by removing two nonadjacent vertices v_1 and v_2 from a snark G, and replacing each edge incident to v_1 or v_2 by semiedges.
ii. $(1, 1)$-semi-graph L' (superedge) is an isolated edge (two semiedges);
iii. $(1, 3, 3)$-semi-graph J' (supervertex) consists of two isolated edges and a vertex;
iv. $(1, 1, 1)$-semi-graph K' (supervertex) consists of a vertex and three semiedges.

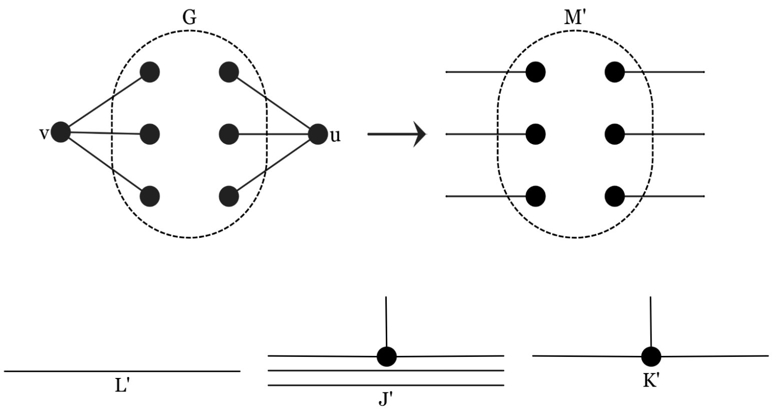

Figure 1. Superedge M' obtained from snark G, superedge L', supervertex J' and supervertex K'.

4. Goldberg snarks

The infinite family of Goldberg snarks was introduced by Goldberg [Goldberg 1981]. The first member of this family, denoted by G_5, has 40 vertices and 60 edges, as illustrated in Figure 2. It is formed by the junction of five semi-graphs referred to as *Goldberg link*, or simply *link*. These snarks grow infinitely and recursively by an odd number of connected links, $\mathcal{L} = 5, 7, 9 \ldots$, with $\mathcal{L} \geq 3$, meaning that G_5 has five links, G_7 has seven links, and so on. The family can be generated iteratively by adding an odd number of links.

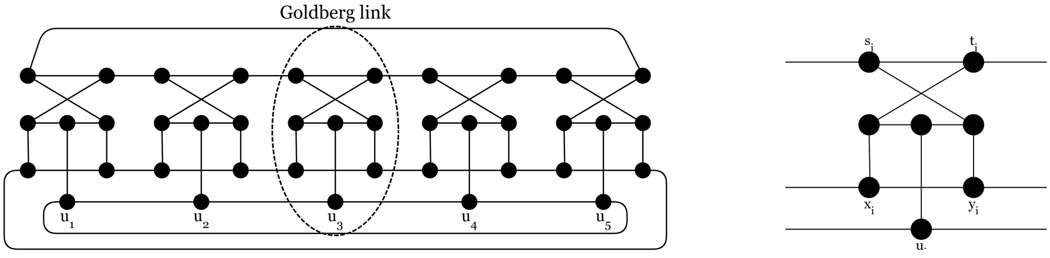

Figure 2. First member of Goldberg snark G_5 and Goldberg link.

5. Our Main Result

In particular, we consider known Type 2 snarks of girth 4, constructed by which we refer to as the *brick's snark*. We use this graph as a superedge, as ilustrated in Figure 3, along with the well-known family of Goldberg snarks, which was proved to be Type 1 by Campos et al. [Campos et al. 2011]. The superedge (ξ) was obtained by removing vertices $v, u \in G$ from brick's snark, as also shown.

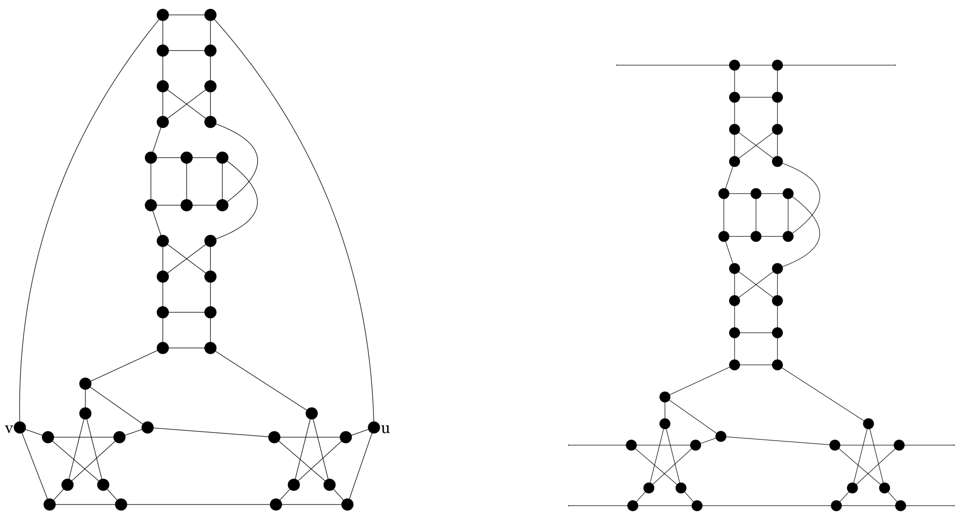

Figure 3. Girth 4 Type 2 snark G and a superedge of G.

We construct a new infinite family of small girth snarks by applying Kochol's superposition, using the superedge ξ in the Goldberg snarks. The superposition occurs as follows: the cycle formed by the vertices u_1, u_2, u_3, u_4 and u_5 constitutes the supervertices, and a superedge ξ is added between each pair of adjacent vertices, which we referred to as the *Golbrick* family. We prove that all members of the Golbrick are Type 2, resulting in Theorem 1. The members of this family are denote by Gb_i, for $i = 5, 7, 9 \ldots$, with the first member being Gb_5.

Theorem 1. *All members of the Golbrick snarks obtained by superposition of the Goldberg snark G_i, for odd $i \geq 5$ (Type 2), with a brick's snark (Type 1) are Type 2.*

Sketch of proof First, we obtain a 4-total coloring for G_5 and for the semi-graph, $\mathcal{L}_2$, formed by two links. At each step, a new $\mathcal{L}_2$ is added to a previous member of the

Goldberg snarks, ensuring that all members admit a 4-total coloring. The same coloring pattern was applied to the Golbrick family. Without loss of generality, colors 1 and 2 were assigned to the supervertices used in the superposition, resulting in a partial coloring of Gb_5, since only the superedge ξ remained uncolored, as illustrated in Figure 4.

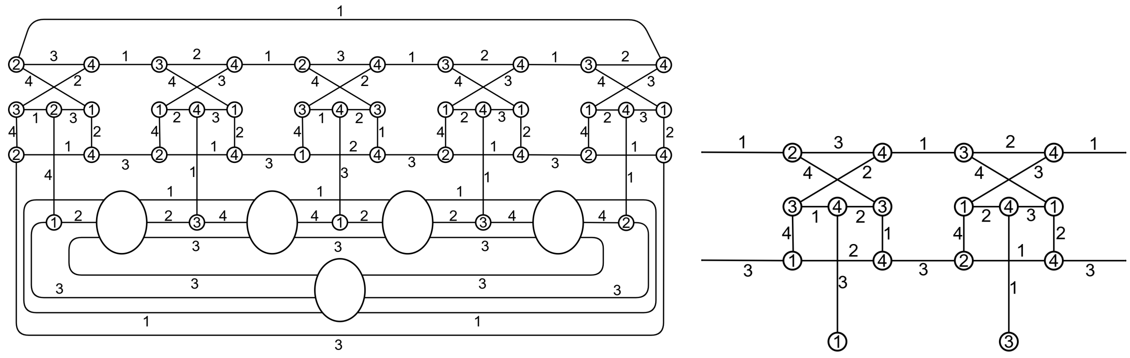

Figure 4. A partial 4-total colorings of Golbrick Gb_5 and of $\mathcal{L}_2$.

The superedge ξ does not admit a 4-total coloring, then all members of the family are Type 2. To achieve a total coloring of this family, we construct three distinct 5-total colorings of the superedges, as shown in Figure 5, ensuring that these colorings propagate across the entire family.

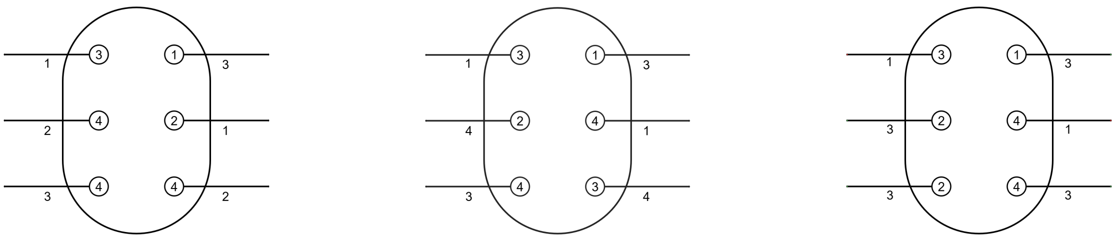

Figure 5. 5-total coloring of superedges ξ_1, ξ_2 and ξ.

6. Conclusion

We have shown that it is possible to apply the Kochol's superposition to obtain an infinite family of Type 2 snarks with small girth.

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