New upper bound for Gallai-Ramsey number of brooms

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Abstract. *Let m, l, and k be integers greater than 1. The broom graph with a handle of length l and m bristles is denoted by $B_{l,m}$. A Gallai coloring of a graph G is an edge coloring of G that does not have a triangle with edges of 3 distinct colors. The k-colored Gallai-Ramsey number of the graph H denoted by $GR_k(H)$ is the smallest natural number n such that every Gallai k-coloring of the complete graph with n vertices contains a monochromatic copy of H. Hamlin proved in 2019 that, for $m \geq 7l/2 + 3$, $GR_k(B_{l,m}) \leq (k-2)(\lceil l/2 \rceil - 1) + 3m - \lceil 3l/2 \rceil - 2$. We prove that if $m \geq 2$ and $l \geq \max\{2m, 5\}$ $GR_k(B_{l,m}) \leq (k-2)(\lceil l/2 \rceil - 1) + 3m + 3\lceil l/2 \rceil - 2$.*

1. Introdução

Let $k \in \mathbb{N}$ and G be a finite simple graph. The k-colored Ramsey number of G, denoted by $R_k(G)$, is the smallest natural number m such that every k-coloring of the edges of K_m contains a monochromatic copy of G. The fact that $R_k(G)$ is well-defined is a theorem from [Ramsey 1930], which established a strong and active research area in Combinatorics, with applications in Information Theory, Geometry, Topology, among other fields (see [Rosta 2004]). The most important open case is when G is a clique. Recently, in [Campos et al. 2023], a substantial improvement in the upper bound was made, where it was proven that there exists $\epsilon > 0$ such that, for all large n, $R_2(K_n) \leq (4-\epsilon)^n$. However, Ramsey numbers have already been studied for a vast number of other graphs (see the dynamic survey [Radziszowski 2012]).

Denote by $B_{l,m}$ the broom with m bristles and handle length l. It is known that:

Theorem 1.1. [Erdős et al. 1982], [Yu and Li 2016]

$$R_2(B_{l,m}) = \begin{cases} m + l + \lceil l/2 \rceil - 1, & \text{if } l \geq 2m - 1; \\ 2(m+l) - 2\lceil l/2 \rceil - 1, & \text{if } 4 \leq l \leq 2m - 2. \end{cases}$$

Let H be a simple graph and σ an (edge) coloring of G. We say that σ contains a rainbow copy of H if there exists a copy of H in G where all edges have distinct colors. A Gallai coloring is an edge coloring of a complete graph such that there is no rainbow copy of a complete 3-vertex graph. A Gallai partition of G is a partition of $V(G)$ into sets $V_1, \ldots, V_p$ with $p \geq 2$ such that at most two colors of σ are used on the edges in $E(G) - (E(V_1) \cup \cdots \cup E(V_p))$, and only one color is used on the edges between any fixed pair (V_i, V_j). In [Gyárfás and Simony 2004] and [Hall et al. 2014], it was shown that:

Theorem 1.2. Let σ be a Gallai coloring of the complete graph G with $|V(G)| \geq 2$. Then the following results hold: [Gyárfás and Simony 2004] There exists a Gallai partition of G. [Hall et al. 2014] Moreover, a minimal Gallai partition (i.e., where p is minimized) is such that, for each color i of σ appearing on edges between the parts, the subgraph of the reduced graph induced by i is connected.

Intersecting both topics above, the k-colored Gallai-Ramsey number of graphs, denoted by $GR_k(G)$, is the smallest natural number m such that every Gallai k-coloring of a complete graph with m vertices contains a monochromatic copy of G. Naturally, it follows that $GR_k(G) \leq R_k(G)$. In [Wu et al. 2019], it was shown that:

Theorem 1.3. [Wu et al. 2019] For a connected bipartite graph H with the size of the smallest part s, we have:

$$GR_k(H) \geq R_2(H) + (s-1)(k-2).$$

Combining the above with Lemma 1.1, it follows that:

Observation 1.4. Let m, l, k be integers, then

$$GR_k(B_{l,m}) \geq \begin{cases} m + l + \lceil l/2 \rceil - 1 + (k-2)(\lceil l/2 \rceil - 1), & \text{if } l \geq 2m-1; \\ 2m + 2\lfloor l/2 \rfloor - 1 + (k-2)(\lceil l/2 \rceil - 1), & \text{if } 4 \leq l \leq 2m-2. \end{cases}$$

Given a Gallai coloring σ of the complete graph G and a positive integer m, the number of colors whose maximum connected component (in the graph induced by the edges of that color) has size at least m is denoted by $q_\sigma(G, m)$. In [Zhang et al. 2022] it was shown that this quantity is related to the Gallai Ramsey number of cycles. Let C_n the cycle graph with n vertices.

Theorem 1.5. [Zhang et al. 2022] Let $m, k \geq 3$ and σ be a Gallai coloring of the complete graph G. If

$$|G| \geq q_\sigma(G, m)(m-1) + m + 1$$

then σ contains a copy of C_{2m}. In particular, $GR_k(C_{2m}) \leq (k-2)(m-1) + 3m - 1$.

Finally, in [Hamlin 2019] it was shown that:

Theorem 1.6. [Hamlin 2019] Let k, l, and m be integers such that $m \geq 7l/2 + 3$, then

$$GR_k(B_{l,m}) \leq (k-2)(\lceil l/2 \rceil - 1) + 3m - \lceil 3l/2 \rceil - 2.$$

Moreover, it was conjectured that:

Conjecture 1.7. [Hamlin 2019] Let m, l, k be integers greater than 1, then

$$GR_k(B_{l,m}) = \begin{cases} m + l + \lceil l/2 \rceil - 1 + (k-2)(\lceil l/2 \rceil - 1), & \text{if } l \geq 2m-1; \\ 2m + 2\lfloor l/2 \rfloor - 1 + (k-2)(\lceil l/2 \rceil - 1), & \text{if } 4 \leq l \leq 2m-2. \end{cases}$$

Here, we consider the case $l \geq 2m$ and prove that:

Theorem 1.8. Let $m \geq 2$, $l \geq \max\{2m, 5\}$, and σ be a Gallai coloring of the complete graph G. If

$$|G| \geq max\{(q_\sigma(G, m + \lfloor l/2 \rfloor) - 2)(\lceil l/2 \rceil - 1), 0\} + 3m + 3\lceil l/2 \rceil - 2$$

then σ contains a monochromatic copy of $B_{l,m}$. In particular, for any positive integer $k > 1$, $GR_k(B_{l,m}) \leq 3m + 3\lceil l/2 \rceil - 2 + (k-2)(\lceil l/2 \rceil - 1)$.

2. Results

Below is a classical result on cycles in bipartite graphs, which will be used at the end of the proof of Theorem 1.8.

Theorem 2.1. [Jackson 1981] Let G be a bipartite graph with parts A and D such that $|N(v)| \geq t$ for every $v \in A$ and $t \leq |D| \leq 2t-2$. Then, G contains as a subgraph all cycles with $2m$ vertices for $1 \leq m \leq \min\{|A|, t\}$.

Proof Sketch of Theorem 1.8. First, observe that if σ has only one color the hypothesis on $|G|$ yields the desired result. With two colors, it follows immediately from Theorem 1.1. So we may assume there are at least three colors. Assume, by contradiction, that there exist a Gallai coloring of a complete graph that satifies the hypotheses of the theorem but does not contain a monochromatic copy of $B_{l,m}$. Among these colorings, let σ a Gallai coloring of a complete graph G such that $q_\sigma(G, m+\lfloor l/2 \rfloor)$ is minimal (by Theorem 1.2, he is at least 1). Now define $q = q_\sigma(G, m+\lfloor l/2 \rfloor)$. A color i in the coloring σ is called special if the subgraph of G induced by edges of color i has its largest connected component of size at least $m+\lfloor l/2 \rfloor$ (i.e., it contributes to the count of q). Assume that the special colors are the first q colors. For a color i in σ, call a set of vertices A i-complete with B if all edges with endpoints in both sets have color i. Now consider subsets $X_1, X_2, \ldots, X_q$ of $V(G)$ and define $X = \bigcup_{i=1}^{q} X_i$ satisfying the following properties: for every $i \in [q]$ X_i is i-complete with $V(G) - X$, and, respecting $|V(G) - X| \geq m + \lfloor l/2 \rfloor$, $|X|$ is maximized. The observation below follows immediately from the definition of these sets.

Observation 2.2. For each $i \in [q]$ $|X_i| \leq \lceil l/2 \rceil - 1$

Let σ' be the coloring induced by σ on the graph $G - X$. Clearly σ' is a k-Gallai coloring of $G - X$ without monochromatic copies of $B_{l,m}$. By Theorem 1.2, let $\mathcal{P}$ be a minimum Gallai partition of $G - X$ with respect to σ', consisting of p elements, denoted as $V_1, V_2, ..., V_p$. Without loss of generality, assume that $|V_1| \leq |V_2| \leq ... \leq |V_p|$. Since $|G - X| \geq m + \lfloor l/2 \rfloor$ the Theorem 1.2 guarantees that all colors appearing on edges between the parts of $\mathcal{P}$ are special. Again, by Theorem 1.2, without loss of generality assume that the only colors that can appear on the edges between these parts are 1 or 2. The observation below follows from this and from maximality of X.

Observation 2.3. $|V_p| \leq m + \lfloor l/2 \rfloor - 1$

Now, let $\mathcal{C}$ be the set of colors that appear on the edges between the parts of $\mathcal{P}$ and let $\mathcal{D} = [q] - \mathcal{C}$. Then observe that

Claim 2.4. If $i \in \mathcal{D}$ then $X_i \neq \emptyset$. In particular, $q \leq 2$.

Proof. On the one hand, let $i \in \mathcal{D}$. Let $k, j \in \mathcal{D}$ be distinct colors. Since σ is a Gallai coloring and the color of the edges between X_t and $G - X$ is t for every $t \in [q]$ it follows that all edges between X_j and X_k are either of color j or color k. Now, assume for contradiction that $X_i = \emptyset$. Then every edge of color i has its endpoints contained either in $\bigcup_{j \in [q]-i} E(G[X_j])$ or $\bigcup_{j \in [p]} E(G[V_j])$ (with $E(G[V])$ denoting the set of edges of G with ends in V if $V \subset V(G)$). From Observations 2.2 and 2.3, it follows that i is not a special color, which is a contradiction that proving the initial claim. With this in mind, assume for contradiction that $q \geq 3$. Since $|\mathcal{C}| \leq 2$ then $q \in \mathcal{D}$. Given this let σ^* be the restriction of σ to $G - X_q$. From the previous part it follows that $q_{\sigma^*}(G, m+\lfloor l/2 \rfloor) \leq q-1$. Furthermore let $j \in [q-1]$ be a special color with respect to σ. At first glance, if $j \in \mathcal{C}$, then j is still special with respect to σ^*, because $X_j \cap G - X = \emptyset$. Moreover, if $j \in \mathcal{D} - \{q\}$ then $X_j \subset G - X_q$. From the first part $X_j \neq \emptyset$ so j is still special with respect to σ^*. Given this it follows that $q_{\sigma^*}(G, m + \lfloor l/2 \rfloor) = q - 1$, with σ^* being a Gallai coloring that does not contain a monochromatic copy of $B_{l,m}$. Finally notice that, by Claim 2.2,

$|G - X_q| \geq |G| - (\lceil l/2 \rceil - 1) = (q-2)(\lceil l/2 \rceil - 1) + 3m + 3\lceil l/2 \rceil - 2 - (\lceil l/2 \rceil - 1) = (q-3)(\lceil l/2 \rceil - 1) + 3m + 3\lceil l/2 \rceil - 2 = (q_{\sigma^*}(G - X_i, m + \lfloor l/2 \rfloor) - 2)(\lceil l/2 \rceil - 1) + 3m + 3\lceil l/2 \rceil - 2$.

Thus, by $q^* \geq 2$ and the minimality of q, it follows that σ^* contains a copy of $B_{l,m}$, a contradiction. □

From this point on let for each $i \in \{1, 2, 3\}$ and $v \in V(G)$, $N_i(v)$ denote the set of neighbors of v in G where the edge with v has color i. Finally observe that

Claim 2.5. $|V_p| \leq \lceil l/2 \rceil - 1$ and $min\{|X_1|, |X_2|\} = 0$.

Proof. For the first part, assume for contradiction that the opposite happens, split G into three parts: V_p, A_1 and A_2 where A_i is the set of vertices of $G - V_p$ that connect to V_p by edges of color i. From here using Observation 2.3 and the pigeonhole principle on A_1 and A_2, it is easy to see that $max\{|A_1|, |A_2|\} \leq \lceil l/2 \rceil + m - 1$, but this implies that $|G| = |V_p| + |A_1| + |A_2| < 3\lceil l/2 \rceil + 3m - 3$ a contradiction. Now assume both are non-empty for contradiction. Then it follows that colors 1 and 2 are the only special colors. By Theorem 1.5, there exists a monochromatic copy of $C_{2\lceil l/2 \rceil}$, C in σ. Since this graph is connected it follows that this copy can only be of color 1 or 2. Without loss of generality, assume it is of color 1. That said i claim that $X_1 = \emptyset$. Let $v \in X_1$. If $v \in V(C)$ then by Observation 2.2 it is easy to see that $|N_1(v) - V(C)| \geq 3m - 1$ and therefore that there exists a copy of $B_{l,m}$ in σ. Otherwise again by Observation 2.2 and the size of the cycle $N_1(v) \cap V(C) \neq \emptyset$. Moreover $|G - V(C)| \geq 3m + \lceil l/2 \rceil - 2$ implies that thus $|N_1(v) - V(C)| \geq 3m - 1$. In both cases a copy of $B_{l,m}$ is found, a contradiction. □

Given the statement above let ρ be a 3-coloring of G such that the edges of $\bigcup_{j \in [q]-i} E(G[X_j])$ and of $\bigcup_{j \in [p]} E(G[V_j])$ have color 3 and preserve the coloring σ on the remaining edges. Note that this coloring is clearly a Gallai coloring and if there exists a monochromatic copy of $B_{l,m}$, it would also exist in σ. Moreover by the same reasoning as before $q = q_\rho(G, m + \lceil l/2 \rceil) = q_\rho(G, \lceil l/2 \rceil)$ and $v, w \in V(G)$ are such that $w \in N_3(v)$, then $N_i(v) = N_i(w)$ for all $i \in \{1, 2, 3\}$. Again, by Theorem 1.5, there exists a monochromatic copy of $C_{2\lceil l/2 \rceil}$ in ρ. Let C and $A = V(C)$ be this copy and set $D = V(G) - A$, so $|D| \geq 3m + \lceil l/2 \rceil - 2$. Fix $v \in A$, then it follows that $|N_1(v) \cap D| \leq m - 1$. Also, observe that, by the choice of ρ, every vertex in $N_1(v)$ is 1-complete with $N_3(v) \cap D$, so $|N_3(v) \cap D| \leq m - 1$. Consequently, $|N_2(v) \cap D| \geq |D| - (2m - 2) \geq m + \lceil l/2 \rceil$. Furthermore note that

$$2(|D| - 2m + 2) - 2 = 2|D| - 4m + 2 \geq |D| \Leftrightarrow |D| \geq 4m - 2.$$

Since $|D| \geq 3m + \lceil l/2 \rceil - 2 \geq 4m - 2$, because $\lceil l/2 \rceil \geq m$, it follows by Theorem 2.1, applied in A and D that there exists a monochromatic copy of $B_{l,m}$ of color 2 in ρ, which, as observed earlier, is a contradiction. This complete the proof of the first part of theorem. In particular, since $q_\sigma(G, m + \lfloor l/2 \rfloor)$ is at most the amount of colours for all Gallai coloring σ of complete graph G, the Theorem 1.8 was proved. □

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