Towards Combinatorial Min-Max Relations from Extended Linear Programming Formulations

Marcel K. de Carli Silva[1*], Mario Leston-Rey[2], Henri M. F. Oliveira[1†], Cristiane M. Sato[2*]

[1] Instituto de Matemática e Estatística, Universidade de São Paulo

[2]Centro de Matemática Computação e Cognição, Universidade Federal do ABC

{henrimichel,mksilva}@ime.usp.br, {c.sato,mario.leston}@ufabc.edu.br

Abstract. *We study the concept of extended total dual integrality and the combinatorial min-max relations that can be derived from this notion. In this work, we prove the extended total dual integrality property for compact extended linear programming formulations for the minimum-cost r-arborescence problem and the maximum-weight perfectly matchable subset problem. From this, we derive dual integer programming problems and show how to solve each pair of primal-dual problems algorithmically and efficiently by adapting classical methods. We also provide a combinatorial interpretation to the dual problem of the minimum-cost r-arborescence problem as a distance splitting problem.*

1. Introduction

Min-max relations stand among the most classic and celebrated results in combinatorial optimization, as evidenced by Schrijver's encyclopedic book [Schrijver 2003]. Most of them are deeply related to the totally dual integral (TDI) property, introduced in [Edmonds and Giles 1977]. Indeed, many min-max relations in [Schrijver 2003] are treated as essentially identical to corresponding TDI claims (see, e.g., Theorems 41.12 and 60.1). Total dual integrality (TDIness) elegantly captures the relationship between primal polyhedron integrality and dual integrality properties. A key consequence is that a TDI formulation determines a polyhedron all of whose vertices are integral.

Linear programming (LP) formulations for combinatorial problems often have exponentially many constraints. While such formulations can still be useful if efficiently separable (and thus amenable to the ellipsoid method), polynomial-sized (or compact) formulations are preferable since they allow for direct solution by practical LP solvers. Introducing additional variables sometimes can yield a compact formulation, remarkably reducing both primal constraints and dual variables to polynomial size. This is called a *compact extended formulation*, as it extends beyond the natural space of variables.

We investigate a notion of total dual integrality for extended formulations introduced in [Martin 1991]. For a system with primary variables x and additional variables y, total dual integrality *with respect to* x (wrt x) still preserves the key consequence of TDIness: ensuring an optimal solution with integral x. The integrality of y is not required

*This work was partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), 408180/2023-4.

†This study was financed, in part, by the São Paulo Research Foundation (FAPESP), Brazil. Process Number #2023/07025-0.

since they are just auxiliary variables. Our research aims to describe new compact extended TDI formulations, interpret the new dual problems, and potentially obtain new combinatorial min-max relations, paired with efficient algorithms.

2. Main Results

Definition 2.1 (see [Martin 1991, de Carli Silva and Tunçel 2020])**.** A system of linear inequalities $Ax + By \le b$, with $A \in \mathbb{Q}^{U\times V}$, $B \in \mathbb{Q}^{U\times W}$, and $b \in \mathbb{Q}^U$, is *totally dual integral wrt* x (x*-TDI*) if, for each vector $c \in \mathbb{Z}^V$, the LP dual to $\max\{\, c^\mathsf{T}x \mid Ax+By \le b\}$ has an integral optimal solution, whenever it has any optimal solution at all.

From Definition 2.1 and [Edmonds and Giles 1977, Theorem (7.1′)], we obtain:

Corollary 2.2. If $Ax + By \le b$ is x-TDI and b is integral, then the projection of the polyhedron defined by $Ax + By \le b$ onto the x-space has only integral vertices.

Below we prove x-TDIness of two compact extended formulations. Corollary 2.2 immediately yields combinatorial min-max relations. However, further investigation is needed to understand natural combinatorial interpretations of the extended dual LPs.

2.1. Minimum-Cost r-Arborescences

Let $D = (V, A)$ be a digraph, and let $r \in V$. A set of arcs $B \subseteq A$ is an r*-arborescence* if, for each $v \in V$, there is exactly one rv-path in (V, B). The *minimum-cost* r*-arborescence* problem is to find an r-arborescence B that minimizes $c(B) \coloneqq \sum_{a\in B} c_a$ for a given cost vector $c \in \mathbb{R}^A_+$. This can be formulated as minimizing $c^\mathsf{T}x$ over $x \in P_{r\text{-arb}}(D)^\uparrow$, the latter being the set of all vectors in $\mathbb{R}^A$ which componentwise dominate some convex combination of incidence vectors of r-arborescences. Known linear inequality systems describing $P_{r\text{-arb}}(D)^\uparrow \subseteq \mathbb{R}^A_+$ in its natural space (one variable for each arc) use exponentially many inequalities (see [Edmonds 1973]). However, $P_{r\text{-arb}}(D)^\uparrow$ can be described compactly as a projection [Maculan 1986, Wong 1984], with an extra variable $f_v \in \mathbb{R}^A$ for each $v \in \overline{r}$:

$$
\begin{array}{llll}
\text{Minimize } c^\mathsf{T}x & & \text{Maximize } \sum_{v\in\overline{r}}(e_v - e_r)^\mathsf{T}y_v & \\
\text{subject to } x \in \mathbb{R}^A_+, & & \text{subject to } \sum_{v\in\overline{r}} z_v \le c, & \\
\quad f_v \in \mathbb{R}^A_+, & \forall v \in \overline{r}, & \quad B_D^\mathsf{T}y_v - z_v \le 0, & \forall v \in \overline{r}, \\
\quad B_D f_v = e_v - e_r, & \forall v \in \overline{r}, & \quad y_v \in \mathbb{R}^V, & \forall v \in \overline{r}, \\
\quad x - f_v \ge 0, & \forall v \in \overline{r}, & \quad z_v \in \mathbb{R}^A_+, & \forall v \in \overline{r}.
\end{array}
\tag{1}
$$

Here, $B_D \in \{0, \pm 1\}^{V\times A}$ is the incidence matrix of D, and $e_v \in \{0, 1\}^V$ is the vector with exactly one non-zero entry at the position indexed by v. Each f_v is a flow of value 1 from r to v; this is encoded by the constraint $B_D f_v = e_v - e_r$. One of our main results is showing this system is x-TDI. Section 3 has an overview of our algorithmic proof.

Theorem 2.3. The system in the primal LP of (1) is x-TDI.

The dual problem in (1) can be interpreted combinatorially as a distance splitting problem that we describe as follows. Any nonnegative cost function $c \in \mathbb{R}^A_+$ gives rise to a distance function $\operatorname{dist}_{(D,c)}\colon V \times V \to [0, \infty]$, where $\operatorname{dist}_{(D,c)}(u, v)$ is the minimum cost wrt c of a uv-path in D. An $\overline{r}$*-splitting* of $c \in \mathbb{R}^A_+$ is a family $\{z_v\}_{v\in\overline{r}} \subseteq \mathbb{R}^A_+$ of nonnegative vectors whose sum does not exceed c, i.e., $\sum_{v\in\overline{r}} z_v \le c$. The dual in (1) is to find an $\overline{r}$-splitting $\{z_v\}_{v\in\overline{r}}$ of c that maximizes $\sum_{v\in\overline{r}} \operatorname{dist}_{(D,z_v)}(r, v)$. By Theorem 2.3, if c is integral, one can range over just integral splittings.

2.2. Maximum-Weight Perfectly Matchable Subset

For a graph $G = (V, E)$ and $U \subseteq V$, if the induced subgraph $G[U]$ has a perfect matching, we say U is *perfectly matchable* and $G[U]$ is a *perfectly matchable subgraph* of G. The *maximum-weight perfectly matchable subset* problem is to find a perfectly matchable subset $U \subseteq V$ maximizing $w(U) := \sum_{u \in U} w(u)$ for a given weight vector $w \in \mathbb{R}^V_+$. The *perfectly matchable subgraph polytope* of G, denoted by $P_{\text{perf. matchable}}(G)$, is the polytope obtained by taking the convex hull of the incidence vectors of all perfectly matchable subsets. If G is bipartite, $P_{\text{perf. matchable}}(G)$ admits a compact extended formulation [Balas and Pulleyblank 1983], using the incidence matrix $B_G \in \{0, 1\}^{V \times E}$ of G, and the vector $\mathbb{1}$ of all ones:

$$\begin{array}{llll} \text{Maximize} & w^\mathsf{T} x & \text{Minimize} & \mathbb{1}^\mathsf{T} v \\ \text{subject to} & x \in \mathbb{R}^V_+, & \text{subject to} & v + u \geq w, \\ & y \in \mathbb{R}^E_+, & & B_G^\mathsf{T} u \leq 0, \\ & x \leq \mathbb{1}, & & v \in \mathbb{R}^V_+, \\ & x - B_G y = 0, & & u \in \mathbb{R}^V. \end{array} \tag{2}$$

Our second main result is to show that this system is x-TDI:

Theorem 2.4. If $G = (V, E)$ is bipartite, the system in the primal LP of (2) is x-TDI.

Our contribution can be seen as an algorithmic reframing of the results in [Balas and Pulleyblank 1983] to the extended TDIness setting, realized via the Hungarian method. In fact, Theorem 2.4 follows from the total unimodularity of B_G, since G is bipartite — a well-known fact also noted in [Balas and Pulleyblank 1983].

3. Overview of Algorithmic Proofs

3.1. Minimum-Cost r-Arborescences

Let $D = (V, A)$ be a digraph, and let $r \in V$. In order to prove Theorem 2.3, by Definition 2.1, we have to prove that, for each $c \in \mathbb{Z}^A_+$, the dual problem in (1) has an integral optimal solution, whenever optimality is achieved. Hence, we may assume that D has an r-arborescence. We adapt an already known algorithm (see [Schrijver 2003, Section 52.1]) so that it also returns an optimal solution for the dual LP in (1). Given as input a digraph $D = (V, A)$, a root $r \in V$, and a cost vector $c \in \mathbb{R}^A_+$, such that D has an r-arborescence, the algorithm outputs a triple $(B, (y_v)_{v \in \overline{r}}, (z_v)_{v \in \overline{r}})$ such that:

(3) (i) $B \subseteq A$ is an r-arborescence in D,
(ii) $\{y_v\}_{v \in \overline{r}} \subseteq \mathbb{R}^V$ and $\{z_v\}_{v \in \overline{r}} \subseteq \mathbb{R}^A_+$ form a feasible solution for the dual LP in (1),
(iii) $c(B) = \sum_{v \in \overline{r}} (e_v - e_r)^\mathsf{T} y_v$;
(iv) moreover, if c is integral, then so are $(y_v)_{v \in \overline{r}}$ and $(z_v)_{v \in \overline{r}}$.

The algorithm is recursive. Let A_0 be the set of arcs with cost zero and set $D_0 := (V, A_0)$. The base case is when D_0 has an r-arborescence B. In this case, it returns $(B, (y_v)_{v \in \overline{r}}, (z_v)_{v \in \overline{r}})$, where $y_v = 0 \in \mathbb{R}^V$ and $z_v = 0 \in \mathbb{R}^A_+$ for each $v \in V$. Next we discuss the recursive step. In this case, since D_0 does not have an r-arborescence, there is a strongly connected component S of D_0, disjoint from the set of vertices reachable from r in D_0, with no arc of A_0 entering S, that is $\delta^{\text{in}}_{D_0}(S) = \varnothing$. However, $\delta^{\text{in}}_D(S) \neq \varnothing$, since

D contains an r-arborescence. We create a new cost vector $\tilde{c}$ by decreasing the cost of every arc in $\delta^{\text{in}}_D(S)$ by $\lambda := \min\{c(a) \mid a \in \delta^{\text{in}}_D(S)\} > 0$. This ensures that $\tilde{c} \geq 0$ and that more arcs will be added to A_0. We obtain the triple $(\tilde{B}, (\tilde{y}_v)_{v\in\bar{r}}, (\tilde{z}_v)_{v\in\bar{r}})$ recursively for the input $(D, r, \tilde{c})$. Next we build $(B, (y_v)_{v\in\bar{r}}, (z_v)_{v\in\bar{r}})$ from $(\tilde{B}, (\tilde{y}_v)_{v\in\bar{r}}, (\tilde{z}_v)_{v\in\bar{r}})$. We must modify $\tilde{B}$ to get B since $\tilde{B}$ may have multiple arcs entering S. So we pick a single arc $b \in \tilde{B}$ from $\delta^{\text{in}}_D(S)$ to keep. Let s be the head of b. We remove all arcs in $\delta^{\text{in}}_D(S) \cup A[S]$ from $\tilde{B}$ except for b, where $A[S]$ denotes the set of arcs of A whose ends are in S. By doing so, we have "broken" our solution, but it is easy to fix it. Since S is strongly connected in D_0, there is an s-arborescence F in the digraph $(S, A_0[S])$ and so we add F to $\tilde{B}$. That is, $B := \big(\tilde{B} \setminus \big(\delta^{\text{in}}_D(S) \cup A[S]\big)\big) \cup \{b\} \cup F$. Finally we set $y_s := \tilde{y}_s + \lambda \mathbb{1}_S \in \mathbb{R}^V$, $z_s := \tilde{z}_s + \lambda \mathbb{1}_{\delta^{\text{in}}_D(S)} \in \mathbb{R}^A_+$ and $y_v := \tilde{y}_v$, $z_v := \tilde{z}_v$ for each $v \in \bar{r} \setminus \{s\}$. Here, $\mathbb{1}_S \in \{0,1\}^V$ is the incidence vector of S and similarly for $\mathbb{1}_{\delta^{\text{in}}_D(S)} \in \{0,1\}^A$. It can be proved that the triple $(B, (y_v)_{v\in\bar{r}}, (z_v)_{v\in\bar{r}})$ satisfies (3).

3.2. Maximum-Weight Perfectly Matchable Subset

In order to prove Theorem 2.4, according to Definition 2.1, we have to prove that, for each $w \in \mathbb{Z}^V_+$, the dual LP in (2) has an integral optimal solution. Our problem is closely related to the maximum-weight matching problem: given a graph $G = (V, E)$ and a weight vector $c \in \mathbb{R}^E_+$, find a matching $M \subseteq E$ of G which maximizes $c(M) := \sum_{e\in M} c(e)$. The Hungarian method is a polynomial-time algorithm that takes as input a bipartite graph $G = (V, E)$ and a weight vector $c \in \mathbb{R}^E$, and returns a pair (M, z), where M is a maximum-weight matching in G, and $z \in \mathbb{R}_+$ is an optimal solution for the LP $\min\{\, \mathbb{1}^\mathsf{T} z \mid B^\mathsf{T}_G z \geq c,\ z \in \mathbb{R}^V_+\}$ with $c(M) = \mathbb{1}^\mathsf{T} z$. Moreover, if w is integral, then so is z. We translate vertex weights $w \in \mathbb{R}^V_+$ to edge weights $c \in \mathbb{R}^E_+$ by assigning the weight of each edge as the sum of weights of its ends. Our algorithm for the maximum perfectly matchable subset problem is the following, where the input is a bipartite graph $G = (V, E)$ and a weight vector $w \in \mathbb{R}^V_+$:

Step 1: Apply the Hungarian method for (G, c) with $c := B^\mathsf{T}_G w$ and obtain (M, z);
Step 2: Define S as the set of vertices covered by M;
Step 3: Define $u := w - z$ and $v := z$, then return (S, u, v).

It is immediate that S is a perfectly matchable subset of V, since M is a perfect matching in $G[S]$. Checking that (u, v) forms a feasible for the dual LP in (2) is straightforward. Furthermore, if w is integral, then z is also integral and so are $u = w - z$ and $v = z$. In order to show the optimality of (u, v), we define a feasible solution for the primal in (2) and derive optimality through weak duality. Set $x := B_G \mathbb{1}_M \in \mathbb{R}^V_+$ and $y := \mathbb{1}_M \in \mathbb{R}^E_+$. Checking that (x, y) form a feasible solution for the primal LP in (2) is straightforward, and then it suffices to show that its objective value satisfies $w^\mathsf{T} x = \mathbb{1}^\mathsf{T} v$.

4. Future directions

Our future directions concerning the two problems we investigated here involve interpreting the dual problems within the min-max relations as combinatorial problems. Although we provided an interpretation for one of the dual problems obtained, other interpretations are possible and may lead to new connections. We also aim to implement the primary research objectives outlined in the introduction, which include developing new compact extended TDI formulations and establishing associated min-max relations for other related problems.

References

Balas, E. and Pulleyblank, W. (1983). The perfectly matchable subgraph polytope of a bipartite graph. In *Proceedings of the symposium on the matching problem: theory, algorithms, and applications (Gaithersburg, Md., 1981)*, volume 13, pages 495–516.

de Carli Silva, M. K. and Tunçel, L. (2020). A notion of total dual integrality for convex, semidefinite, and extended formulations. *SIAM J. Discrete Math.*, 34(1):470–496.

Edmonds, J. (1973). Edge-disjoint branchings. In *Combinatorial algorithms (Courant Comput. Sci. Sympos. 9, New York Univ., New York, 1972)*, pages 91–96. Algorithmics Press, New York.

Edmonds, J. and Giles, R. (1977). A min-max relation for submodular functions on graphs. In *Studies in integer programming (Proc. Workshop, Bonn, 1975)*, Ann. Discrete Math., Vol. 1, pages 185–204. North-Holland.

Maculan, N. (1986). A new linear programming formulation for the shortest s-directed spanning tree problem. *J. Combin. Inform. System Sci.*, 11(2-4):53–56.

Martin, R. K. (1991). Using separation algorithms to generate mixed integer model reformulations. *Oper. Res. Lett.*, 10(3):119–128.

Schrijver, A. (2003). *Combinatorial optimization. Polyhedra and efficiency*, volume 24 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin.

Wong, R. T. (1984). A dual ascent approach for Steiner tree problems on a directed graph. *Math. Program.*, 28(3):271–287.