Decompositions of graphs into trees with bounded degree*

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Abstract. *A decomposition of a graph G is a set of edge-disjoint subgraphs of G that covers the edge set of G. In 1968, Gallai conjectured that every connected simple graph on n vertices can be decomposed into at most $\lceil n/2 \rceil$ elements, in which all elements are paths. In 1977, Chung verified a weaker version of this conjecture, showing that every graph on n vertices admits a decomposition into trees of size at most $\lceil n/2 \rceil$. In 2019, Botler strengthened Chung's result by showing that every graph on n vertices can be decomposed into trees with maximum degree at most $\lceil n/2 \rceil$, while keeping a decomposition size of at most $\lceil n/2 \rceil$. In this paper, we improve Botler's result by reducing the maximum degree to $n/4 + 2$.*

Resumo. *Uma decomposição de um grafo G é um conjunto de subgrafos aresta-disjuntos de G que cobre o conjunto de arestas de G. Em 1968, Gallai conjecturou que todo grafo simples conexo com n vértices pode ser decomposto em no máximo $\lceil n/2 \rceil$ elementos, nos quais todos os elementos são caminhos. Em 1977, Chung verificou uma versão mais fraca dessa conjectura, mostrando que todo grafo com n vértices admite uma decomposição em árvores de tamanho no máximo $\lceil n/2 \rceil$. Em 2019, Botler fortaleceu o resultado de Chung ao verificar que todo grafo com n vértices pode ser decomposto em árvores com grau máximo de no máximo $\lceil n/2 \rceil$, mantendo um tamanho de decomposição de no máximo $\lceil n/2 \rceil$. Neste artigo, melhoramos o resultado de Botler reduzindo o grau máximo para $n/4 + 2$.*

1. Introduction

A *decomposition* of a graph G is a collection $\mathcal{D} = \{H_1, \dots, H_k\}$ of edge-disjoint subgraphs of G whose union covers all edges of G. If H_i is a simple path for every $i \in [k]$, then we say that $\mathcal{D}$ is a *path decomposition* of G. While much progress has been made in understanding path decompositions, a related problem considers minimizing the number of paths required to cover a graph. This measure, introduced by Erdős (see [Lovász 1968]), is known as the *path number* of G, and is denoted by $\text{pn}(G)$. In order to answer Erdős's problem, Gallai posed the following conjecture.

Conjecture 1 (Gallai, 1968)**.** *If G is a connected graph on n vertices, then $pn(G) \leq \lceil n/2 \rceil$.*

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Conjecture 1 was highlighted by Bondy [Bondy 2014] as one of the most beautiful conjectures in Graph Theory. Despite being widely studied, Conjecture 1 remains unsolved. Chung [Chung 1978] investigated a weaker version of Conjecture 1 and proved the following result.

Theorem 2. *If G is a connected graph on n vertices, then G admits a decomposition into at most $\lceil n/2 \rceil$ trees.*

Botler [Botler 2019] proved a stronger version of Theorem 2, where they found a decomposition also into $\lceil n/2 \rceil$ trees but ensured that maximum degree is at most $\lceil n/2 \rceil$. For presenting the following result, we say that a decomposition $\mathcal{D}$ of a graph G is a (k, d)*-decomposition* if each element of $\mathcal{D}$ is a tree with diameter at most k and maximum degree at most d and $|\mathcal{D}| \leq \lceil n/2 \rceil$.

Theorem 3. *Let G be a connected graph on n vertices. Then G admits a $(4, \lceil n/2 \rceil)$-decomposition.*

Note that reducing maximum degree moves us closer to verifying Gallai's conjecture. In this work, we present an alternative proof of Theorem 3, which allows us to improve the result by reducing the maximum degree to $n/4+2$, paving the way for further improvements. At a certain point in the alternative proof, we deal with subgraphs of $K_{2,2}$, specifically controlling the maximum degree of the trees. We noticed that if we worked with $K_{2,\ldots,2}$, we would obtain a lower maximum degree. Thus, in this paper, we work with subgraphs of $K_{2,2,2}$ to achieve the desired maximum degree.

2. Preliminaries

Before our main result, we state the theorem that characterizes all n for which K_n admits a K_3-decomposition.

Theorem 4 (Kirkman)**.** *The complete graph K_n admits a K_3-decomposition if and only if $n \equiv 1, 3 \pmod 6$.*

Botler proved the following lemma that we use in this paper.

Lemma 5. *Let G be a connected graph with at least three vertices. Then, there are two vertices $x, y \in V(G)$ such that $G - x - y$ is connected and either*

i) $d_G(x) = d_G(y) = 1$ *and* $dist_G(x, y) = 2$*; or*
ii) $xy \in E(G)$.

For presenting Lemma 6, we introduce the following notation. Let G be a graph on n vertices and suppose that n is even. Given a path P of G, we denote by $\mathrm{ends(P)}$ the endvertices of P. Let $\mathcal{V} = \{V_1, \ldots, V_{n/2}\}$ be a set of disjoint pairs of vertices of G. We say that $\mathcal{V}$ is a *connected pair partition* if there is, in G, a set of edge-disjoint paths with length 1 or 2 denoted by $\mathcal{P} = \{P_1, \ldots, P_{n/2}\}$ such that $\mathcal{V} = \big\{\mathrm{ends(P_i)} : \mathrm{i} \in [\mathrm{n/2}]\big\}$ Moreover, when G admits a connected pair partition induced by a set of paths $\mathcal{P}$, we say that $\mathcal{P}$ is a *skeleton* of G. As a consequence of Lemma 5, we obtain the following lemma.

Lemma 6. *Let G be a connected graph with n vertices, and suppose that $n \geq 4$. Then G admits a connecting pair partition.*

3. Decomposition into trees with maximum degree at most $n/4 + 2$

In this section, we aim to find a $(6, n/4+2)$-decomposition of G. To achieve this, the following definition ensures that the decomposition has specific properties designed to provide greater control during proofs.

Definition 7. *Let $G \subseteq K_{2,\dots,2}$, and let $\mathcal{V} = \{V_1, \dots, V_k\}$ be the vertex partition of G, where $V_i = \{x_i, y_i\}$ for $i \in [k]$. We say that $\mathcal{F} = \{F_1, \dots, F_k\}$ is a $\mathcal{V}$-*useful *forest decomposition if, for each $i \in [k]$, the following hold:*

i) F_i is a forest with at least two components, F_i^1, F_i^2, and $x_i \in V(F_i^1)$, $y_i \in V(F_i^2)$;

ii) $dist_{F_i^1}(x_i, w) \leq 2$ (resp. $dist_{F_i^2}(y_i, w) \leq 2$) for every $w \in V(F_i^1)$ (resp. $w \in V(F_i^2)$).

Now, we present an example of a $\mathcal{V}$-useful forest decomposition.

Definition 8. *Let $G \subseteq K_{2,\dots,2}$, and let $\mathcal{V} = \{V_1, \dots, V_k\}$ be the vertex partition of G, where $V_i = \{x_i, y_i\}$ for $i \in [3]$. The* canonical decomposition forward *of $K_{2,2,2}$ is the collection of forests $\mathcal{F} = \{F_1, F_2, F_3\}$, where each F_i is defined as follows (indices are taken modulo 3)*

$$F_i = \{T_i^1, T_i^2 \colon T_i^1 = E(G) \cap \{x_i x_{i+1}, x_{i+1} y_{i+2}\}, T_i^2 = E(G) \cap \{y_i y_{i+1}, y_{i+1} x_{i+2}\}\},$$

Note that, in a canonical decomposition forward, V_{i+1} is the only part that can have vertices with degree greater than 2 in F_i. Therefore, we say that F_i is *centered* on V_{i+1}. Moreover, we say that F_i is *constructed* from V_i and, although F_i is constructed from V_i, it is not mandatory for the edges $x_i x_{i+1}$ and $y_i y_{i+1}$ to exist. Also, we define *canonical decomposition backward* in a similar way to canonical decomposition forward, but with indices taken as -1 modulo 3 instead of $+1$ modulo 3.

Now, we present a stronger version of Definition 7. For the following definition, we denote $\omega^*(G)$ as the number of non-trivial components of G, that is, the components that are not isolated vertices.

Definition 9. *Let $\mathcal{F} = \{F_1, \dots, F_k\}$ be a $\mathcal{V}$-useful forest decomposition of $K_{2,\dots,2}$. We say that a $\mathcal{F}$ is a $\mathcal{V}$-*ideal *forest decomposition if there is a bijection $f \colon [k] \to [k]$ such that $\omega^*(F_i) = \sum_{w \in V_{f(i)}} d_{F_i}(w)$. Furthermore, the part $V_{f(i)}$ is called* main leaves *of F_i.*

The goal of the $\mathcal{V}$-ideal forest decomposition is to ensure that for each i, the part $V_{f(i)}$ is the set of main leaves of precisely one forest F_i. Canonical decompositions are defined to be an initial template, and although canonical decompositions are $\mathcal{V}$-useful, it may not be $\mathcal{V}$-ideal. The main challenge arises when, after defining F_1, F_2, and F_3, the same V_i must be designated as the set of main leaves for two distinct forests.

In our paper, we present a computational proof to find a $\mathcal{V}$-ideal forest decomposition of any subgraph of $K_{2,2,2}$. We believe that a mathematical proof of Lemma 10 can also be achieved using an auxiliary bipartite graph and Hall's theorem.

Lemma 10. *Let $G \subseteq K_{2,2,2}$, and let $\mathcal{V}$ be its vertex partition. Then G admits a $\mathcal{V}$-ideal forest decomposition.*

With the results of Lemma 10 and Theorem 4, we are able to prove a stronger version of Theorem 3 for a specific case.

Theorem 11. *If n is even and $n/2 \equiv 1, 3 \pmod 6$, then every graph on n vertices admits a $(6, \lceil n/4 \rceil)$-decomposition.*

Proof. Let G be a connected graph on n vertices such that $n/2 \equiv 1, 3 \pmod 6$. By Lemma 6, G admits a connecting pair partition $\mathcal{V}$ induced by $\mathcal{P}$ of $V(G)$. Define G' as the graph obtained by removing the edges of the paths $P_i \in \mathcal{P}$ from $E(G)$, i.e., $V(G') = V(G)$ and $E(G') = E(G) \setminus \bigcup_{i=1}^{n/2} E(P_i)$. Now, let $\mathcal{V} = \bigcup_{d \in [n/2]} V_d$ and assume $V_d = \{x_d, y_d\}$ for each $d \in [n/2]$. By construction, the n vertices are partitioned into $n/2$ parts $V_1, \ldots, V_{n/2}$. Let G^* be the complete auxiliary graph obtained from G' by considering the parts as vertices, i.e., the vertex $v_d \in V(G')$ represents the part V_d. Since $G^* \approx K_{n/2}$ and by Theorem 4, G^* admits a triangle decomposition when $n/2 \equiv 1, 3 \pmod 6$, we can partition $E(G^*)$ into triangles such that each pair of vertices cannot be in more than one triangle. Since $d_{G^*}(v_i) \leq d_{K_{n/2}}(u) = (n/2) - 1$, we have that v_i is in at most $(n/2 - 2)/2 = (n/4) - 1$ triangles, for any $i \in [n/2]$. Therefore, considering only the triangles in which v_i is present and considering the part $V_i(G')$, there are at most $(n/4) - 1$ subgraphs of $K_{2,2,2}$ to which V_i belongs. By Lemma 10, there exists a $\mathcal{V}$-ideal forest decomposition for each of these subgraphs of $K_{2,2,2}$. By Definition 9, $V_i(G')$ is the set of main leaves of exactly one forest in each of the $(n/4) - 1$ decompositions of the subgraphs of $K_{2,2,2}$. Let F_i be the union of the $n/4 - 1$ forests in which $V_i(G')$ is the set of main leaves. Since the only common vertices of the forest that compose F_i are x_i and y_i, F_i is also a forest. Since $\mathcal{V}$ is a connecting pair partition, either $x_i y_i \in E(G)$ or there is a vertex $w \in V(G)$ for which $x_i w, y_i w \in E(G)$. In both cases no edge of $E(P_i)$ is in F_i. Now, let $T_i = F_i \cup E(P_i)$. That is, T_i is a tree in which x_i and y_i are the vertices with maximum degree, $\lceil n/4 \rceil - 1 + 1 = \lceil n/4 \rceil$. Finally, since $\text{dist}_{T_i}(x_i, y_i) \leq 2$, $\text{dist}_{F_i}(x_i, v)$, $\text{dist}_{F_i}(y_i, w) \leq 2$ for all $v, w \in V(F_i')$, then $\text{diam}(T_i) \leq 2 + 2 + 2 = 6$. Thus, G admits a $(6, \lceil n/4 \rceil)$-decomposition. □

Observe that when we return the edges of $\mathcal{P}$ we could create cycles. In fact, our computational proof ensures this does not happen. We chose not to present such a technicality in order to ease readability of this paper.

Finally, note that Theorem 11 requires $n \equiv 2, 6 \pmod{12}$. Now given a graph G, by adding a path P with at most seven new vertices, and adding an edge joining one of its endvertices to any arbitrary vertex of G, we obtain a graph G' for which $n \equiv 2, 6 \pmod{12}$. Then, by applying Theorem 11 to G' we obtain a tree decomposition in which the extra trees are contained in P, and then can be removed to obtain the desired tree decomposition of G.

Corollary 12. *Every graph on n vertices admits a $(6, \lceil n/4 \rceil + 2)$-decomposition.*

4. Concluding remarks

In this paper, we simplify the proof of Theorem 3 and reduce the bound on the maximum degree to $\lceil n/4 \rceil + 2$. Observe that reducing the upper bound to 2 amounts to proving Conjecture 1. Future work includes presenting a proof of Lemma 10 without the use of computers, reducing the bound to $\lceil n/6 \rceil$, and possibly to $\sqrt{n}$.

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