Ramsey Goodness of paths versus $K_{3,t}$*

Fábio Botler[1], Luiz Paulo Freire Moreira [2], João Pedro de Souza [3]

[1]Universidade de São Paulo

[2]Universidade Federal de Pernambuco

[3]Colégio Pedro II
Universidade Federal do Rio de Janeiro

fbotler@ime.usp.br, luiz.fmoreira@ufpe.br, jpsouza@cos.ufrj.br

Abstract. *Given graphs G and H, we say that G is H-*good *if the Ramsey number $R(G,H)$ equals the trivial lower bound $(|G|-1)(\chi(H)-1)+\sigma(H)$, where $\chi(H)$ denotes the usual chromatic number of H, and $\sigma(H)$ denotes the minimum size of a color class in a $\chi(H)$-coloring of H. In 2013, Allen, Brightwell, and Skokan conjectured that P_n is H-good for every $n \geq \chi(H)|H|$. A result of Pokrovskiy and Sudakov (2017) implies that this conjecture holds when $\chi(H) \geq 4$. We study the case $\chi(H) = 2$ and show that P_n is H-good for $n \geq 2 \cdot |V(H)|$, with $H \subseteq K_{3,t}$.*

Resumo. *Dado grafos G e H, dizemos que G é H-*good *se o número de Ramsey $R(G,H)$ for igual ao limitante inferior trivial $(|G|-1)(\chi(H)-1)+\sigma(H)$, onde $\chi(H)$ denota o número cromático usual de H, e $\sigma(H)$ denota o tamanho mínimo de uma classe de cor em uma $\chi(H)$-coloração de H. Em 2013, Allen et al. conjecturaram que P_n é H-good para todo $n \geq \chi(H)|H|$. Um resultado de Pokrovskiy e Sudakov (2017) implica que tal conjectura vale quando $\chi(H) \geq 4$. Nós estudamos o caso $\chi(H) = 2$ e provamos que P_n é H-good para todo $n \geq 2 \cdot |V(H)|$, com $H \subseteq K_{3,t}$.*

1. Introduction

All graphs in this paper are finite, undirected, and simple. The *Ramsey number* $R(G,H)$ is the smallest positive integer N such that every red–blue coloring of the edges of the complete graph with N vertices K_N contains a red copy of G or a blue copy of H. When this is the case, we write $K_N \to (G,H)$. Ramsey theory is a fundamental area of combinatorics (see [Conlon et al. 2015, Botler et al. 2022]). Burr [Burr 1981] observed the following lower bound. Let $\sigma(H)$ be the smallest class size in a proper $\chi(H)$-coloring of H, where $\chi(H)$ denotes the usual chromatic number of H. If G is a connected graph with $|G| \geq \sigma(H)$, where $|G|$ denotes the number of vertices in G, then

$$R(G,H) \geq (|G|-1)(\chi(H)-1)+\sigma(H).$$

*This research has been partially supported by Coordenação de Aperfeiçoamento de Pessoal de Nível Superior – Brasil – CAPES – Finance Code 001. F. Botler is supported by CNPq (304315/2022-2) and CAPES (88887.878880/2023-00). L. Moreira is supported by FAPESB (APP0044/2023). CNPq is the National Council for Scientific and Technological Development of Brazil. FAPESB is the Bahia Research Foundation.

Indeed, let $N = (|G| - 1)(\chi(H) - 1) + \sigma(H) - 1$ and consider the red-blue coloring of $E(K_N)$ obtained from $\chi(H) - 1$ disjoint red cliques of size $|G| - 1$ and one red clique of size $\sigma(H) - 1$, by coloring the remaining edges blue. Such a coloring avoids a red copy of G because every red component has at most $|G| - 1$ vertices, and avoids a blue copy of H because the graph induced by the blue edges is a $\chi(H)$-partite graph whose smallest class has size less than $\sigma(H)$.

For a fixed graph H, we say that a graph G is H*-good* if $R(G, H) = (|G| - 1)(\chi(H) - 1) + \sigma(H)$. The concept of Ramsey goodness, introduced by Burr and Erdős [Burr and Erdős 1983], has been extensively studied for various graph classes (see [Allen et al. 2013, Chvátal 1977, Nikiforov and Rousseau 2009, Pokrovskiy and Sudakov 2020, Balla et al. 2018, Moreira 2021]). In this work, we study the Ramsey goodness of paths. More specifically, we are interested in determining for which n the path P_n is H-good. Since every graph is a subgraph of a complete multipartite graph, it suffices to explore the case H is a complete multipartite graph.

Erdős et al. [Erdős et al. 1985] proved that there is a function f for which P_n is H-good whenever $n \geq f(|H|)$. Allen, Brightwell, and Skokan [Allen et al. 2013] conjectured that P_n is H-good whenever $n \geq \chi(H) \cdot |H|$. Recently, Pokrovskiy and Sudakov [Pokrovskiy and Sudakov 2017] proved that P_n is H-good whenever $n \geq 4 \cdot |H|$, settling Allen et al.'s conjecture in the case $\chi(H) \geq 4$. Moreover, they presented an example that shows that the constant 4 cannot be lowered below 2, i.e., that P_n is not $K_{s,t}$-good if $n < 2t$. On the other hand, we proved [Botler et al. 2024] that for any $\varepsilon > 0$, P_n is H-good whenever $n \geq (2 + \varepsilon)|H|$ and H satisfies a certain imbalance condition (that depends on ε). In this paper we show a family of bipartite graphs for which the constant 4 can be replaced by 2.

Theorem 1. *If* $n = 6 + 2t$*, then* P_n *is* $K_{3,t}$*-good.*

2. Main result

In this section we prove Theorem 1. For that we use the following lemmas. For a proof of Lemma 1 see [Botler et al. 2024, Lemma 2].

Lemma 1. *Let* s, t *be two positive integers with* $s \leq t$*, and let* G *be a graph obtained from an independent set* X *and a graph with vertex set* Y *with* $X \cap Y = \emptyset$*, for which* $e(X, Y) = 0$*. If* $|X| \geq s$ *and* $|X| + |Y| \geq s + t$*, then* $\overline{G}$ *contains a* $K_{s,t}$*, whose part of size* s *is in* X*.*

Lemma 2. *Let* s, t *be positive integers and* $s \leq t$*. Let* G *be a graph on* $3s + 2t - 1$ *vertices,* $n = 2(s + t)$*, and let* P *and* C *be, respectively, a longest path and longest cycle of* G*. If* $|C| + 1 = |P|$*, then* G *has a* P_n *or* $\overline{G}$ *has a* $K_{s,t}$*.*

Proof. Let G, n, P and C be defined as in the statement. If G has a P_n, then we are done. Suppose $|P| < 2(s+t)$. Let $C = u_1 \cdots u_m u_1$. Let X the set of vertices of $G \setminus C$ that have neighbors in C, and $Z = G \setminus (V(C) \cup X)$. Note that $|X| + |Z| \geq s$. Since $|C| + 1 = |P|$, the set X is an independent set in G, and moreover, no vertex of X has a neighbor in Z. In what follows, we prove that $\overline{G \setminus Z}$ contains a copy K' of $K_{s-|Z|,t}$, such that the part of size $s - |Z|$ of K' is in X. Then we can use Z to obtain the desired copy of $K_{s,t}$. Observe that if u_i has a neighbor v in X, then u_{i-1} and u_{i+1} have no neighbor in X, otherwise G contains a cycle of order $|C| + 1$ or a path or order $|C| + 2 = |P| + 1$, a contradiction. Therefore, there is a set $C' \subseteq C$ with at least $\lceil |C|/2 \rceil$ vertices that have no neighbors in X. Since G has no P_{2s+2t}, we have $\lceil |C|/2 \rceil \le (|C|+1)/2 < s+t$, and hence from $3s+2t-1 = |G| = \lceil |C|/2 \rceil + \lfloor |C|/2 \rfloor + |X| + |Z|$, we obtain $\lceil |C|/2 \rceil + |X| \ge s+t-|Z|$. Thus, by Lemma 1, $\overline{G \setminus X}$ contains the desired copy of $K_{s-|Z|,t}$. □

In our proof, we use a variation of the Posá extension-rotation technique. In short, this technique identifies candidates for endvertices of a path through consecutive rotations (see, e.g., [Brandt et al. 2006]). In this work, we explore the structure exposed by one fixed rotation (which we call lollipop below), obtaining a lower bound on the number of candidates for endvertices in terms of the number of vertices that are nonadjacent to all original endvertices. A *lollipop* in a graph G is a subgraph $L = C \cup P$ of G in which $C = u_0u_1 \cdots u_m u_0$ is a cycle and $P = v_1 \cdots v_\ell$ is a path for which $V(C) \cap V(P) = \emptyset$ and $u_0v_1 \in E(G)$. Given a lollipop as above, we consider the following sets.

- $BB = \{w \in V(L) : u_1w, u_mw, v_\ell w \notin E(G)\}$
- $B_1 = \{u_i : u_{i+1}v_\ell \in E(G)\}$
- $B_2 = \{v_i : v_{i+1}u_1 \in E(G) \text{ or } v_{i+1}u_m \in E(G)\}$
- $B_3 = \{v_i : v_{i-1}v_\ell \in E(G)\}$
- $B_{out} = B_1 \cup B_2 \cup B_3 \cup \{u_1, u_m, v_\ell\}$

Intuitively, BB is a set of vertices that can complete u_1, u_m, v_ℓ to a blue $K_{3,|BB|}$; and B_1, B_2, B_3 are new candidates for endvertices obtained from one more rotation. Note also that $u_2, u_{m-1}, u_0, v_{\ell-1} \notin BB$.

Lemma 3. *Let $t \ge 3$ be a positive integer. Let $n = 2t+6$ and G be a graph on $2t+8$ vertices, and suppose that G contains a lollipop containing a longest path of G. Let $L = P \cup C$ be such a lollipop in which C has minimum size, and let BB, B_1, B_2, B_3, and B_{out} be defined as above. If $P_n \not\subset G$, then we have $K_{3,t} \subset \overline{G}$ or $|BB| + |B_{out}| \ge |L| - 4$.*

Proof. The idea is to show that all but at most four vertices of L contribute to the sum $|BB| + |B_{out}|$. By the minimality of C we can assume that u_1 and u_m do not have other neighbors than u_0, u_2 and u_{m-1} inside C. We claim that for each $i \in [m] \setminus \{0, 2, m-1\}$, we have $u_i \in BB$ or $u_{i-1} \in B_{out}$. Indeed, if $u_i \in V(C) \setminus \{u_2, u_0, u_{m-1}\}$ is such that $u_i \notin BB$, then $u_iv_\ell \in E(G)$, and hence $u_{i-1} \in B_1 \subseteq B_{out}$. Now, we consider the vertices of P. If there is $i \in [\ell] \setminus \{\ell-1\}$ for which $u_1v_{i+1} \in E(G)$ and $v_{i-1}v_\ell \in E(G)$, then $C' = v_{i-1} \cdots v_1u_0u_m \cdots u_1v_{i+1} \cdots v_\ell v_{i-1}$ of order $|L|-1$. By Lemma 2, we have $K_{3,t} \subset \overline{G}$, as desired. The case in which $u_mv_{i+1} \in E(G)$ and $v_{i-1}v_\ell \in E(G)$ is analogous. Thus, we may assume that for every $i \in [\ell] \setminus \{\ell-1\}$, we have $v_{i-1}v_\ell \notin E(G)$, or $u_1v_{i+1} \notin E(G)$ and $u_mv_{i+1} \notin E(G)$. In the former case $v_i \in B_2$, and in the latter case $v_i \in B_3$. Therefore, $v \in B_{out}$ for every $v \in V(P) \setminus \{v_{\ell-1}\}$. This concludes the proof. □

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Fix a red–blue coloring of K_N, and let G be the graph induced by the red edges of K_N. We may assume that G contains no copy of $P_n = P_{6+2t}$, and $\overline{G}$ has no copy of $K_{3,t}$. Suppose that G contains a lollipop containing a longest path of G, and let $L = P \cup C$ be such a lollipop in which C has minimum size, and let BB and B_{out} be defined as above. Then, by Lemma 3, $|BB| + |B_{out}| \ge |L| - 4$.

Let $O = V(G) \setminus V(L)$. Since $|L| < n$ we have $|O| \ge 3$. If $|L| = 1$, then G is empty, and there is $K_{3,t} \subseteq \overline{G}$. If there is an edge joining u_1, u_m or v_ℓ to a vertex in O, then we can find a path of order $|L|+1$ in G, a contradiction. If $|BB|+|O| \geq t$, then we pick $\{u_1, u_m, v_\ell\}$ as one of the parts of $K_{3,t}$, and $BB \cup O$ as the other part, a contradiction. Thus, we may assume that $|BB|+|O| \leq t-1$. If there is an edge joining a vertex in B_{out} to a vertex in O, then we can find a path of order $|L|+1$ in G, a contradiction. Since $|L| = N - |O|$, we have

$$|B_{out}| \geq |L| - 4 - |BB| = N - |O| - 4 - |BB| \geq N - 4 - (t-1) = t+5. \quad (1)$$

Since $|O| \geq 3$, we can pick three vertices in O and t vertices in B_{out} to obtain the desired $K_{3,t} \subset \overline{G}$, a contradiction.

Thus we may assume that G does not contain a lollipop containing a longest path of G. Let $P = u_1 \cdots u_r$ be a longest path in G and let $O = V(G) \setminus V(P)$. Then, the only neighbors of u_1 and u_r are, respectively, u_2 and u_{r-1}. Suppose that u_2 has a neighbor u in O and let $P' = uu_2 \cdots u_r$. By the maximality of P, u has no neighbor in O. Also, u cannot be a neighbor of any other vertex w of P, otherwise we could find a lollipop with the vertices of P'. In this case, we choose the t vertices of $V(G) \setminus \{u, u_1, u_2, u_{r-1}, u_r\}$ and $\{u, u_1, u_r\}$ to obtain $K_{3,t} \subseteq \overline{G}$, a contradiction. Therefore u_2 has no neighbors in O. Let $A = \{u_1, u_2, u_3, u_{r-1}, u_r\}$ and $B = V(P) \setminus A = \{u_4, \ldots, u_{r-2}\}$. If u_2 has at most $t-1$ neighbors in B, then u_2 is not adjacent to at least

$$|O| + |B| - (t-1) = |O| + |P| - |A| - (t-1) = N - t - 4 = 2t + 8 - t - 4 = t + 4$$

vertices. Thus, we can pick t vertices of $\big(V(G) \setminus N_P(u_2)\big) \setminus A$ and $\{u_1, u_2, u_r\}$ to obtain a $K_{3,t}$ in $\overline{G}$ because u_1 and u_r are only adjacent to vertices in A, a contradiction.

Thus, we may assume that u_2 has at least t neighbors in B. Suppose that for some neighbor $u_i \in B$ of u_2, the vertex u_{i-1} has a neighbor $u \in O$. As above, u has no neighbors in $V(G) \setminus \{u_{i-i}\}$. In this case, we choose t vertices in $V(G) \setminus \{u, u_1, u_2, u_{r-1}, u_r\}$ and $\{u, u_1, u_r\}$ to obtain $K_{3,t} \subseteq \overline{G}$, a contradiction. Then, we may assume that u_{i-1} has no neighbors in O. Let $K = \{u_{i-1} : u_i \in N_P(u_2) \cap B\}$ be the set of "predecessors" of $N_P(u_2) \setminus \{u_1, u_3, u_{r-1}, u_r\}$. Note that $|K| = |N_P(u_2) \cap B| \geq t$. Since $|P| \leq n-1$, we have $|O| \geq 3$. Since there is no edge between O and K, we can choose three vertices in O and t vertices in K to obtain the desired $K_{3,t} \subset \overline{G}$, a contradiction. This concludes the proof. □

We can prove a similar result for $K_{2,t}$ with $n = 2(t+2)$ and $N = 2t+5$. First, note that, by Equation (1), we could get $K'_N \to (P'_n, K_{3,t})$ in Theorem 1, even if we put $N' = n' - 1 = N$. If we could not create a lollipop from a longest path of the graph G induced by the red edges of K_N, then we could easily pick the two end vertices of any longest P and t other vertices that are not neighbors of them to obtain the desired $K_{2,t}$.

3. Future work

In this paper, we have shown that if $n \geq 2(t+3)$, then P_n is $K_{3,t}$-good. We believe that we can extend the techniques used to generalize Theorem 1 for $K_{4,t}$, $K_{5,t}$, by generalizing the idea of lollipop, but new insights are necessary to deal with $K_{s,t}$, with $s \geq 6$. For example, the case in which we can find precisely the endvertices s in the longest path P can be proved using the Posá rotation extension technique (see [Brandt et al. 2006]), but the generalizations of BB and B_{out} so far did not fit well. We expect to prove a generalization of Lemma 3 providing more rotations to the endvertices.

References

Allen, P., Brightwell, G., and Skokan, J. (2013). Ramsey-goodness—and otherwise. *Combinatorica*, 33:125–160.

Balla, I., Pokrovskiy, A., and Sudakov, B. (2018). Ramsey goodness of bounded degree trees. *Combinatorics, Probability and Computing*, 27(3):289–309.

Botler, F., Moreira, L., and de Souza, J. P. (2024). Ramsey goodness of paths and unbalanced graphs. *arXiv preprint arXiv:2410.19942*.

Botler, F. H., Collares, M., Martins, T., Mendonça, W., Morris, R., and Mota, G. O. (2022). *Combinatória*. IMPA.

Brandt, S., Broersma, H., Diestel, R., and Kriesell, M. (2006). Global connectivity and expansion: long cycles and factors in F-connected graphs. *Combinatorica*, 26:17–36.

Burr, S. A. (1981). Ramsey numbers involving graphs with long suspended paths. *Journal of the London Mathematical Society*, 2(3):405–413.

Burr, S. A. and Erdős, P. (1983). Generalizations of a Ramsey-theoretic result of Chvátal. *Journal of Graph Theory*, 7(1):39–51.

Chvátal, V. (1977). Tree-complete graph Ramsey numbers. *Journal of Graph Theory*, 1(1):93–93.

Conlon, D., Fox, J., and Sudakov, B. (2015). Recent developments in graph Ramsey theory. *Surveys in combinatorics*, 424(2015):49–118.

Erdős, P., Faudree, R. J., Rousseau, C. C., and Schelp, R. H. (1985). Multipartite graph—sparse graph Ramsey numbers. *Combinatorica*, 5:311–318.

Moreira, L. (2021). Ramsey goodness of clique versus paths in random graphs. *SIAM Journal on Discrete Mathematics*, 35(3):2210–2222.

Nikiforov, V. and Rousseau, C. C. (2009). Ramsey goodness and beyond. *Combinatorica*, 29:227–262.

Pokrovskiy, A. and Sudakov, B. (2017). Ramsey goodness of paths. *Journal of Combinatorial Theory, Series B*, 122:384–390.

Pokrovskiy, A. and Sudakov, B. (2020). Ramsey goodness of cycles. *SIAM Journal on Discrete Mathematics*, 34(3):1884–1908.