An XP Approximation Scheme for AFTP in 2D*

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Abstract. *The Angular Freeze-Tag Problem (AFTP) concerns the distribution of mission data within a satellite swarm. In this scenario, satellites cannot be reached simultaneously via a single broadcast; instead, highly focused directional antennas are required. As a result, a satellite can only transmit when its antenna is oriented toward the recipient. This alignment may require reorienting the antenna, incurring a time cost proportional to the rotation angle. The objective is to minimize the makespan, i.e., the total distribution time.*
Approximating AFTP in the plane within a factor better than $5/3$ is known to be NP-hard. We overcome this bound by introducing two new parameters: the smallest orientation change needed for a satellite's antenna, starting from its initial position, to align with another satellite, and the total energy (rotation angle). This parametrization enables us to design a parameterized approximation scheme that runs in XP time. Furthermore, our approach extends to the AFTP variant that aims to minimize total energy instead of makespan.

1. Introduction

Many optimization problems arise in swarm robotics. The Angular Freeze-Tag Problem (AFTP), introduced by Fekete and Krupke [Fekete and Krupke 2018], models data broadcast within a satellite swarm using peer-to-peer communication. In space missions, instructions are hard to deliver due to signal loss from omnidirectional broadcasts, instead requiring precisely aimed directional antennas, which demand time-consuming maneuvers. To speed up transmission, satellites that have received the data can relay it to others, forming intricate communication trees. AFTP simplifies this scenario by assuming static agents, instantaneous transmission, and ray-like signals. The objective is to minimize the total distribution time (makespan), assuming that each satellite can adjust its antenna only after receiving the data, which starts with a single source. Thus, unlike typical network problems based on distance, AFTP focuses on minimizing angular adjustment costs.

In their work, Fekete and Krupke proved a $5/3$ lower bound of approximation and gave a 9-approximation for the 2D case, using a result by Beck and Newman on the linear search problem [Beck and Newman 1970]. They also proposed a mixed integer program (MIP) for AFTP in arbitrary dimensions. To our knowledge, this is

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the only paper on AFTP. In contrast, the original Freeze-Tag Problem (FTP), introduced by Arkin et al. [Arkin et al. 2002], has received more attention [Arkin et al. 2003, Sztainberg et al. 2004, Brunner and Wellman 2020, Pedrosa and de Oliveira Silva 2023]. FTP involves activating robots by moving active ones to inactive ones, aiming to minimize the time of the last activation. While it uses distance instead of angle, FTP is also NP-hard in many domains, but in contrast, it admits a PTAS in the 2D plane [Arkin et al. 2006].

Although it is technically possible to use a directional transmission antenna with an omnidirectional receiver, this setup is inefficient, likely requiring two antennas per satellite. A more practical approach extends AFTP by requiring both ends to adjust their orientations. In his PhD thesis, Krupke [Krupke 2022] introduced this variant, the Bidirectional Angular Freeze-Tag Problem (BAFTP). Using a different construction involving visibility-obstructing obstacles, he proved the same $5/3$ approximation lower bound for BAFTP. In this model, all antennas can rotate from the start, but activation requires both sides to orient toward each other.

From the perspective of parameterized algorithms, the goal is typically to find exact solutions with running times that may include a non-polynomial factor depending only on an input parameter. More formally, a parameterized decision problem with parameter k is *fixed-parameter tractable* (FPT) if it can be decided in time $f(k) \cdot n^{O(1)}$, where n is the input size and f is a computable function of k. Similarly, a problem is called *slice-wise polynomial* (XP) if it can be decided in time $f(k) \cdot n^{g(k)}$, where g is another computable function of k. For a more comprehensive exposition of parameterized algorithms and complexity, the reader may refer to the book by Cygan et al. [Cygan et al. 2015].

In this work, we bypass the $5/3$ approximation lower bound for AFTP in 2D by presenting a parameterized approximation scheme that runs in XP time. Also, our algorithm applies to the variant that minimizes total energy (rotation angle) instead of makespan.

2. Problem Definition

Consider a set $P = \{p_1, \dots, p_n\}$ of distinct positions in the Euclidean plane. Each $p_i \in P$ corresponds to a satellite and is associated with an angle α_i, i.e., its antenna's initial orientation angle. A satellite can either be *active* or *inactive*, and initially, only p_1 is active. Active satellites can freely rotate their antennas, while inactive ones cannot. An inactive satellite p_i is *activated* by an active satellite p_j if it aims its antenna towards p_i. The time incurred to rotate an antenna is proportional to the change in its angle.

A solution is a *schedule* of rotations that activate all satellites, denoted by S. For each satellite $p_i \in P$, the schedule contains a *starting direction* d_i, either *clockwise* or *counterclockwise*, and a sequence of positive rotation angles $S_i = (s_{i,1}, \dots, s_{i,k_i})$, where $k_i \geq 0$ is the length of S_i. This sequence describes the rotations that are carried out by p_i once activated as follows: starting at orientation angle α_i, the satellite rotates its antenna by $s_{i,1}$ in direction d_i, then by $s_{i,2}$ in the opposite direction, and so on. Notice that $s_{i,j}$ corresponds to the change in the orientation angle at step j, and the rotation direction alternates between clockwise and counterclockwise. See Figure 1 for an example of an AFTP instance and a corresponding solution. The *makespan* of a schedule S, denoted by $M(S)$, is defined as the time at which the last satellite is activated. Also, the *total energy* of a schedule S, denoted by $E(S)$, is the sum of the total rotation (in radians) done by all satellites. We consider two problems depending on the objective function. For AFTP, the

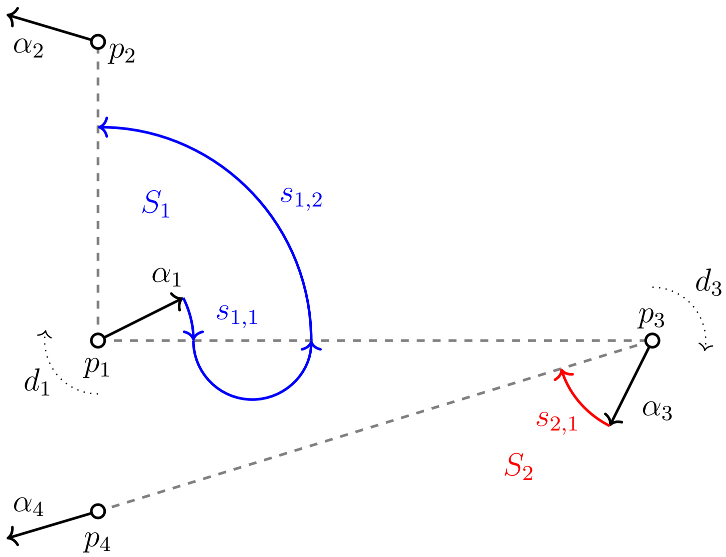

Figure 1. AFTP instance example. In an optimal solution, p_1 first activates p_3, which then activates p_4, while p_1 simultaneously turns back to activate p_2.

objective is to minimize $M(S)$, while for the Total Energy Angular Freeze-Tag Problem (E-AFTP), the goal is to minimize $E(S)$.

A solution is called *rational* if: (i) at any moment, each active satellite making an activation rotates its antenna towards some other inactive satellite in the shortest direction, and no two active satellites claim to activate the same inactive satellite; and (ii) if every inactive satellite is already claimed, then active satellites with no target stop rotating their antennas. From now on, we assume that every optimal solution is rational.

3. An XP Approximation Scheme for 2D

Given an instance of AFTP or E-AFTP, let δ be the smallest angle a satellite's antenna must rotate from its initial orientation to point to any other satellite. We refer to δ as the instance's *minimum angle*. Note that for every S_i of positive length in a rational schedule S, it holds that $s_{i,1} \geq \delta$. Assuming $\delta > 0$, we present an approximation scheme for AFTP in 2D that runs in XP time, parameterized by the total energy and $\lceil 1/\delta \rceil$.

Theorem 1. *Consider an instance I of AFTP in 2D of minimum angle $\delta > 0$. Let E be a real number and k a positive integer. There is an algorithm that runs in time $(n\frac{Ek}{\delta})^{O(\frac{Ek}{\delta})}$ and either proves that every optimal solution requires more than E total energy, or else finds a solution of makespan at most $(1 + 1/k)$OPT(I).*

Proof. We will say that a sequence is μ-discrete if all its values are integer multiples of $\mu = \frac{\delta}{4k}$. Moreover, we call a solution μ-discrete if all its sequences are μ-discrete. We discretize the rotation of antennas into steps with sizes multiples of μ and concentrate on μ-discrete solutions. The following lemma guarantees that this restriction has a minimal impact on the makespan or total energy of optimal solutions.

Lemma 1. *Any rational solution S to I can be transformed into a μ-discrete solution S^μ so that $M(S^\mu) \leq (1 + 1/k)M(S)$ and $E(S^\mu) \leq (1 + 1/k)E(S)$.*

Next, we show how to determine whether or not there is a feasible μ-discrete solution of total energy at most $(1+{}^1\!/\!_k)E$. If there is such a solution, we find one with optimal makespan, and because of Lemma 1, the makespan will be at most $(1 + {}^1\!/\!_k)\text{OPT}(I)$. If not, we conclude that every optimal solution for the original problem has a total energy of more than E.

Given a candidate solution, checking whether it is feasible and computing its makespan and total energy can be easily done in polynomial time. If an optimal solution for I has total energy at most E, we can find an optimal μ-discrete solution by exhaustive enumeration: After each rotation, a satellite's antenna can be in one of $O({}^E\!/\!_\mu)$ valid orientation angles, and, in each step, it transitions between two such orientations. Thus, there are $O({}^{E^2}\!/\!_{\mu^2})$ possible transitions per satellite, and the total number of transitions for all satellites is $O(n\frac{E^2}{\mu^2})$. Amongst this whole set of transitions, we can choose at most $O({}^E\!/\!_\mu)$ of them if we are to respect the total energy limit of $(1 + {}^1\!/\!_k)E(S)$. So, there are

$$\binom{O(n\frac{E^2}{\mu^2})}{O(\frac{E}{\mu})}$$

candidate solutions. Using Stirling's approximation, this number can be bounded by $(n\frac{E}{\mu})^{O(\frac{E}{\mu})}$. Therefore, we can do all the required computations in $(n\frac{Ek}{\delta})^{O(\frac{Ek}{\delta})}$ time. □

We observe that both considered parameters appear together in the running time. By defining $R := \frac{E}{\delta}$, which we call the *aspect ratio* of the instance, the running time of the algorithm can be rewritten as $(nRk)^{O(Rk)}$. For E-AFTP, we get a better bound as the total energy of an optimal solution never exceeds 2π (a solution in which p_1 makes a full rotation achieves such a bound). Using the previous algorithm with minor modifications, we get the following:

Theorem 2. *For every positive integer k, there is a parameterized $(1 + {}^1\!/\!_k)$-approximation algorithm for E-AFTP in 2D, for instances with minimum angle $\delta > 0$, that runs in $(n\frac{k}{\delta})^{O(\frac{k}{\delta})}$ time.*

However, we currently lack any proof of hardness for E-AFTP in 2D, and this problem might as well be solvable in polynomial time.

4. Conclusion

Although AFTP is a Euclidean problem, it lacks a metric structure. Specifically, it violates two metric axioms: (i) there are infinitely many points a satellite antenna can reach without rotation (i.e., all points along its current direction), and (ii) symmetry is lost, as the rotation needed for satellite p_i to align its antenna with satellite p_j is not necessarily the same as for p_j to do the same with p_i.

Despite the challenges, we successfully presented a new algorithmic result for AFTP. Handling small angles was a central difficulty, particularly where large and small angles coexist. Hence, our small-angle assumption is reasonable, as arbitrarily precise motion is physically unattainable. However, from a theoretical standpoint, it requires further justification. The natural next step is to improve our approximation scheme to achieve an FPT running time instead of an XP one and to determine the computational complexity of E-AFTP.

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