On the Hamiltonicity of Timbral Graphs

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Abstract. *We study the existence of a Hamiltonian cycle in timbral graphs, presenting a complete characterization of which timbral graphs are Hamiltonian.*

1. Introduction

We study the existence of a Hamiltonian cycle in timbral graphs $T_{n,k,\ell}$. This class of graphs was defined by [Akhmedov and Winter 2014], who proved that all timbral graphs with $n \geq 3$ are Hamiltonian. We present an alternative proof of the Hamiltonicity of $T_{n,k,\ell}$ for $n \geq 3$, relying on algebraic properties and connectivity of timbral graphs. In addition, we solve the case $n = 2$, starting by showing that $T_{2,k,\ell}$ is not Hamiltonian when $\ell = 0$ or $k - \ell$ is even. For the remaining cases, we provide an inductive construction to establish the existence of Hamiltonian cycles.

2. Timbral gaphs

In what follows, n, k and ℓ are nonnegative integers such that $n \geq 2$ and $k \geq \ell + 1$. The *timbral* graph $T_{n,k,\ell}$ is the graph whose vertex set is the set of all words of length k built over an alphabet of n symbols and two vertices are adjacent if and only if they agree, exactly, in ℓ coordinates. Timbral graphs were defined by [Akhmedov and Winter 2014] as a model for a very specific way of composing music, based on the enumeration of musical elements through Hamiltonian cycles. Thus, the main property examined by Akhmedov and Winter was precisely the existence of a Hamiltonian cycle in timbral graphs. Regarding this property, they applied a theorem of [Gravier 1997] to prove that $T_{n,k,\ell}$ contains a spanning subgraph that has a Hamiltonian cycle. Their proof, however, is restricted to the case where $n \geq 3$ and they left the case $n = 2$ open, considering it more complex. However, noting that timbral graphs are Cayley graphs, we can reduce the proof of Hamiltonicity of timbral graphs to a proof of connectivity for these graphs. This is particularly possible due to the well-known fact, independently proved by [Chen and Quimpo 1981] and [Marušič 1983], that every connected Cayley graph of an abelian group of order at least three is Hamiltonian.

We would also like to note that the Hamiltonicity of timbral graphs is related to combinatorial Gray codes, since a Hamiltonian cycle in $T_{n,k,\ell}$ is precisely a cyclic sequence of words of length k where two subsequent words agree in exactly ℓ coordinates, which aligns with the definition of a combinatorial Gray code [Savage 1997].

3. Algebraic Preliminaries

Our study of the existence of Hamiltonian cycles in timbral graphs is based on the notion of a Cayley graph, which is an algebraic property of a graph. Thus, before arriving at our main results, we first review some suitable algebraic notions.

Definition 1. *A set A endowed with a binary operation $* : A \times A \to A$ is a* **group** *if $*$ is associative, has a neutral element, and every element of A has an inverse element in A. If the operation is also commutative, then A is said to be an* abelian group.

This definition is particularly important because the vertex set $\mathbb{Z}_n^k$ of a timbral graph, which is the Cartesian product of k copies of $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$, is an abelian group under vector addition modulo n.

Definition 2. *A graph G is a* **Cayley graph** *if there exists an operation $*$ on $V(G)$ and a subset $S \subset V(G)$ such that $V(G)$ endowed with $*$ is a group and, for each $u \in V(G)$, the neighborhood of u, written $N(u)$, is the set $\{u * v : v \in S\}$.*

Equivalently, we can construct the Cayley graph of a group A with a generating set $S \subset A$ — which does not contain the neutral element of A— as the graph whose vertex set is A and whose edge set is $\{\{a, a * s\} : a \in A, s \in S\}$.

Since $\mathbb{Z}_n^k$ is an abelian group, it suffices to examine the neighborhood of each vertex to prove that $T_{n,k,\ell}$ is a Cayley graph. Now, let us define $S_{n,k,\ell}$ as the set of elements of $\mathbb{Z}_n^k$ that have exactly $k - \ell$ nonzero coordinates. Each element of $S_{n,k,\ell}$ represents a change in exactly $k - \ell$ coordinates, which is exactly what an edge represents in $T_{n,k,\ell}$. Thus, we can obtain the neighborhood of a vertex u by adding u with every element in $S_{n,k,\ell}$.

Proposition 1. *$T_{n,k,\ell}$ is a Cayley graph.*

Next, we will see how the algebraic property of being a Cayley graph can help us characterize the Hamitonian timbral graphs.

4. Hamiltonian cycles in Timbral Graphs

[Akhmedov and Winter 2014] applied properties of the tensor product of graphs to prove that $T_{n,k,\ell}$, is Hamiltonian, whenever $n \geq 3$. We present an alternative proof of this result and also solve the case $n = 2$, which was an open case. The following theorem summarizes our main results.

Theorem 1. *The only non-Hamiltonian timbral graphs are the $T_{2,k,\ell}$, when $\ell = 0$ or $k - \ell$ is even.*

Since $T_{n,k,\ell}$ is a Cayley graph of $\mathbb{Z}_n^k$, based on a result proved by [Chen and Quimpo 1981] and independently by [Marušič 1983], we can prove the existence of a Hamiltonian cycle in a timbral graph as a consequence of its connectivity.

Theorem 2. [Chen and Quimpo 1981, Marušič 1983] *Every connected Cayley graph of an abelian group with at least three vertices is Hamiltonian.*

From Proposition 1, we know that $T_{n,k,\ell}$ is a Cayley graph and, thus, we get the following corollary.

Corollary 1. *$T_{n,k,\ell}$ is Hamiltonian if and only if it is connected and $n^k \geq 3$.*

When $n = 2$, we find several examples of disconnected timbral graphs. However, when admitting $n \geq 3$, we can always build a path between any two vertices in the graph.

Lemma 1. *$T_{n,k,\ell}$ is connected, for all $n \geq 3$.*

To solve the case $n = 2$, we first observe that every graph of the form $T_{2,k,0}$ is non-Hamiltonian, since its edge set is a matching — it is either disconnected or the $K_2 = T_{2,1,0}$. Moreover, this is not the only type of disconnected timbral graph.

We define the *parity of a binary sequence* as the parity of the sum of its terms. For example, 01101 is an odd sequence, since it sums to 3. Now observe that, by flipping an even number of bits of a binary sequence, we always preserve its parity, as the numbers of flipped zeros and ones must match in parity. Thus, since any edge in $T_{2,k,\ell}$ represents a flip on $k - \ell$ bits, there is no path between vertices of distinct parities when $k - \ell$ is even.

Lemma 2. *$T_{2,k,\ell}$ is disconnected whenever $k - \ell$ is even or $\ell = 0$.*

When $k - \ell$ is odd, on the other hand, any edge represents a change in the parity, leading to a bipartition. In this case, it is also possible to prove that the graph is connected and, consequently, Hamiltonian. However, we would rather give an inductive proof for a Hamiltonian cycle in the binary timbral graph with odd $k - \ell$.

We must prove that the graph $T_{2,(2m-1)+\ell,\ell}$ is Hamiltonian, for all $m, \ell \in \mathbb{N}$ such that $m \geq 1$ and $\ell \geq 1$. Since our induction is on ℓ, we begin by proving that $T_{2,2m,1}$ is Hamiltonian for all $m \geq 1$. This serves as our base case, which we also prove by induction, now on m. For this purpose, we introduce a few key definitions.

Definition 3. *Let $r \in \{1, 2, \ldots, k\}$. Let $u \in \mathbb{Z}_2^k$ and $x \in \mathbb{Z}_2^r$. We say that x is a* **prefix** *of u if $u_i = x_i$, for all $i \in \mathbb{Z}_r$.*

Now we can use prefixes to partition the vertex set of $T_{2,k,\ell}$.

Definition 4. *Let $r \in \{1, 2, \ldots, k\}$ and $\alpha \in \mathbb{Z}_2^r$. We define X_α as the set of even vertices of $T_{2,k,\ell}$ that have α as a prefix. Analogously, we define Y_α as the set of odd vertices of $T_{2,k,\ell}$ that have α as a prefix.*

For example, take the graph $T_{2,4,1}$. In this case, we have $X_{00} = \{0000, 0011\}$ and $Y_{00} = \{0001, 0010\}$. Equivalently, the prefix 11 induces the sets $X_{11} = \{1100, 1111\}$ and $Y_{11} = \{1101, 1110\}$. Note that both sets $X_{00} \cup Y_{11}$ and $X_{11} \cup Y_{00}$ induce subgraphs isomorphic to the graph $T_{2,2,1}$, which is the C_4. In general, this exemplifies the fact that, by taking a prefix α and its complement $\bar{\alpha}$ — the word obtained by flipping every bit in α — the set $X_\alpha \cup Y_{\bar{\alpha}}$ in $T_{2,2m,1}$ induces a copy of the graph $T_{2,2(m-1),1}$. We formalize it with the following lemma.

Lemma 3. *Let $m \geq 2$ be a positive integer. The subgraph of $T_{2,2m,1}$ induced by the set $X_\alpha \cup Y_{\bar{\alpha}}$ is isomorphic to $T_{2,2(m-1),1}$, for all $\alpha \in \mathbb{Z}_2^2$.*

This result allows us to divide $T_{2,2m,1}$ into four copies of $T_{2,2(m-1),1}$. By applying the inductive hypothesis and linking the cycles in the subgraphs, we construct a Hamiltonian cycle in the full graph.

Lemma 4. *$T_{2,2m,1}$ is Hamiltonian, for all $m \geq 1$.*

As we mentioned, Lemma 4 proves the basis of our main induction. Now, to prove the Hamiltonicity of $T_{2,2m-1+\ell,\ell}$, we partition the graph into two subgraphs isomorphic to $T_{2,2m-2+\ell,\ell-1}$.

Lemma 5. *Let* $G = T_{2,2m-1+\ell,\ell}$*. The sets* $V_0 = \{v \in V(G) : v_0 = 0\}$ *and* $V_1 = \{v \in V(G) : v_0 = 1\}$ *induce subgraphs isomorphic to* $T_{2,2m-2+\ell,\ell-1}$*.*

Thus, we can use the inductive hypothesis to obtain a cycle in $T_{2,2m-2+\ell,\ell-1}$ and, by taking its copies in each subgraph, build a Hamiltonian cycle in the full graph.

Lemma 6. $T_{2,2m-1+\ell,\ell}$ *is Hamiltonian, for all nonnegative integers* m, ℓ *such that* $m \geq 1$ *and* $\ell \geq 1$*.*

Now, we have closed the proof of Theorem 1, splitting it into three lemmas. First, Lemma 1 shows that $T_{n,k,\ell}$ is Hamiltonian for all $n \geq 3$. Regarding the binary case, Lemma 2 states that $T_{2,k,\ell}$ is disconnected — thus, non-Hamiltonian — if $k - \ell$ is even or $\ell = 0$. Then, Lemma 6 ensures that the remaining case, which is $T_{2,k,\ell}$ when $\ell \geq 1$ and $k - \ell$ is odd, is Hamiltonian. Thus, the only timbral graphs which are not Hamiltonian are the $T_{2,k,\ell}$, when $k - \ell$ is even or $\ell = 0$.

5. Perspectives

We present a proof of the Hamiltonicity of timbral graphs $T_{n,k,\ell}$ for $n \geq 3$, based on their algebraic structure as Cayley graphs. In addition, we solve the open case $n = 2$, fully characterizing non-Hamiltonian timbral graphs as those with $\ell = 0$ or $k - \ell$ is even. For the remaining cases, we provide an inductive construction to establish the existence of Hamiltonian cycles.

A natural extension of this work would be a deeper investigation of Hamiltonian cycles in timbral graphs. This includes finding the number of Hamiltonian cycles in $T_{n,k,\ell}$ and analyzing the structural properties of the cycles derived from our inductive construction. An additional goal would be to design algorithms to find Hamiltonian cycles with specific properties, based on the theory of combinatorial Gray codes.

References

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