New bound on the odd chromatic number of planar graphs with maximum degree at most 4*

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Abstract. *An* odd coloring *of a graph G is a proper vertex coloring where, for every vertex $v \in V(G)$, there is a color that appears an odd number of times in $N_G(v)$. The smallest k for which G admits an odd coloring is the* (proper) odd chromatic number *and it is denoted $\chi_o(G)$. This coloring was introduced in 2022, and it was conjectured that $\chi_o(G) \leq 5$ for every planar graph G, while the best known upper bound is $\chi_o(G) \leq 8$. It is also known that any graph G satisfies $\chi_o(G) \leq \lfloor 3\Delta(G)/2 \rfloor + 2$. This paper shows that every planar graph G with $\Delta(G) \leq 4$ has $\chi_o(G) \leq 7$.*

1. Introduction

Let G be a simple graph. A *vertex k-coloring* of G is an assignment $\phi\colon V(G) \to \{1, \ldots, k\}$. It is called *proper* when $\phi(u) \neq \phi(v)$ for any $uv \in E(G)$. An *odd k-coloring* of G is a proper vertex k-coloring such that, for every $u \in V(G)$, there is a color that appears an odd number of times in $N_G(u)$. The smallest k for which there is an odd k-coloring of G is called the *odd chromatic number*, and it is denoted $\chi_o(G)$. Clearly, all graphs have an odd coloring and $\chi_o(G) \leq |V(G)|$.

[Petruševski and Škrekovski 2022] introduced the odd coloring motivated by odd edge colorings of graphs and some frequency assignment problems. They conjectured the following, which, if true, would be the best possible, since $\chi_o(C_5) = 5$.

Conjecture 1 ([Petruševski and Škrekovski 2022]). *For any planar graph G, $\chi_o(G) \leq 5$.*

They also proved a first upper bound for χ_o in planar graphs, showing that $\chi_o(G) \leq 9$. This result was later improved by [Petr and Portier 2023].

Theorem 1 ([Petr and Portier 2023]). *For any planar graph G, $\chi_o(G) \leq 8$.*

In the following, let G be a planar graph. [Cho et al. 2023] showed that for $c \geq 5$, if G has girth at least $\lceil 4c/(c-2) \rceil$, then $\chi_o(G) \leq c$; if G has girth at least 11, then $\chi_o(G) \leq 4$; and if G has girth at least 6, then $\chi_o(G) \leq 6$. [Kashima and Zhu 2024] showed that if G is outerplanar, then $\chi_o(G) \leq 5$. We say that two cycles are *adjacent* if they share an edge and are *intersecting* if they share a vertex. Also, a cycle of length x is called an *x-cycle*. [Miao et al. 2024] showed that if G has girth ate least 4 and does not contain intersecting C_4s, then $\chi_o(G) \leq 5$. Moreover, they proved the following result.

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Theorem 2 ([Miao et al. 2024]). *For any planar graph G without adjacent triangles, $\chi_o(G) \leq 7$.*

It is worth mentioning that several other results on odd colorings appeared after [Petruševski and Škrekovski 2022]'s seminal paper, not all of them considering planar graphs [Caro et al. 2022]. In fact, [Caro et al. 2022] conjectured the following.

Conjecture 2 ([Caro et al. 2022]). *For any graph G, $\chi_o(G) \leq \Delta(G) + 1$.*

We are interested in planar graphs of bounded maximum degree. The best-known result that considers maximum degree states that for any graph G, we have $\chi_o(G) \leq \lfloor 3\Delta(G)/2 \rfloor + 2$ [Dai et al. 2024].

The previous result shows that if G is a planar graph with $\Delta(G) \leq 4$, then $\chi_o(G) \leq 8$, which does not improve on [Petr and Portier 2023]'s result. In this paper, we show the following.

Theorem 3. *For any planar graph G with $\Delta(G) \leq 4$, $\chi_o(G) \leq 7$.*

Our proof, sketched in the next section, mixes some combinatorial results together with an application of the *discharging method* [Cranston and West 2017]. This method has been very useful for proving results on planar graphs. Indeed, its most famous application is in the proof of the Four Color Theorem.

2. Proof sketch of Theorem 3

To prove Theorem 3, first we prove an auxiliary lemma that gives insight into the structure of a minimal counterexample to our result (Lemma 1). Then, using the Discharging Method, we prove that these structures should actually exist (Lemma 2). Combining both gives us that there is no minimal counterexample to our theorem, thus it must be true.

First, we need some more definitions. A vertex u is called an *ℓ-vertex* if $d_G(u) = \ell$ and it is called an *ℓ^+-vertex* (resp. *ℓ^--vertex*) if $d_G(u) \geq \ell$ (resp. $d_G(u) \leq \ell$). If G is a plane graph, we denote the *length* of a face f as $d_G(f)$ and the set of all faces as $F(G)$. Similarly to the notation for vertices, a face $f \in F(G)$ is called an *ℓ-face* if $d_G(f) = \ell$ and it is called an *ℓ^+-face* (resp. ℓ^--face) if $d_G(f) \geq \ell$ (resp. $d_G(f) \leq \ell$).

Let ϕ be an odd coloring for G. We say a vertex u is *critical in ϕ* if only one color appears an odd number of times in its neighborhood. A *witness* for a vertex u in ϕ is a color that appears an odd number of times in u's neighborhood; we denote it by $wit(u)$ when u is a critical vertex. A vertex u is *happy in ϕ* if it has at least one witness in ϕ.

Lemma 1. *If G is a minimal counterexample to Theorem 3, then G does not contain:*

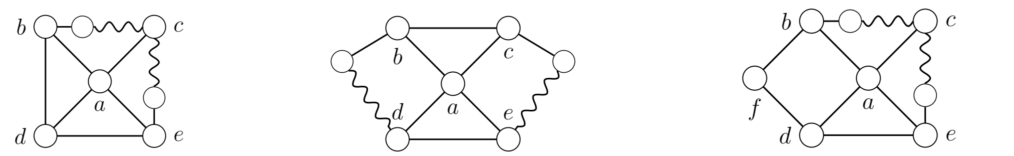
1. *cut edges or cut vertices;*
2. *1-, 2- or 3-vertices;*
3. *adjacent or intersecting 3-faces, and;*
4. *a 3-face adjacent to a 4-face.*

Proof sketch. Let G be a planar graph with the least number of vertices for which $\Delta(G) \leq 4$ and $\chi_o(G) > 7$. Suppose, by contradiction, that G contains intersecting 3-faces, as in Figure 1b. Let G' such that $V(G') := V(G) \setminus \{a\}$ and $E(G') := (E(G) \cup \{be, ce\}) \setminus \{de\}$ By the choice of G and because G' is planar, we know that $\chi(G') \leq 7$. Let ϕ' be an odd 7-coloring of G'. Build an odd 7-coloring ϕ for G in the following way. Let

$\phi(u) = \phi'(u)$ for all $u \in V(G) \setminus \{a, d\}$. If $\phi'(d) = \phi'(e)$, then $\phi(d)$ needs to receive a different color; choose one avoiding the colors of its neighbors and its neighbors' witnesses (at most 4 different colors) to keep the coloring proper and odd. Now vertex a is happy in ϕ, since the colors $\phi(b)$, $\phi(c)$, and $\phi(e)$ differ. To color a and keep the coloring proper and odd, the color $\phi(a)$ should not be any color in the set $X = \{\phi(b), \phi(c), \phi(d), \phi(e), wit(b), wit(c), wit(d), wit(e)\}$. If $|X| \leq 6$, then there is a color available to be $\phi(a)$ and we are done. Otherwise, $|X| = 7$ and there is only one color that appears in two elements of X. In this case, if $|\{\phi(b), \phi(c), \phi(d), \phi(e)\}| = 4$, then choose $x \in \{b, c, d, e\}$ such that x is critical and $\phi(x)$ does not appear in another element of X; assign $\phi(a) = \phi(x)$ and recolor x with another color, which is possible since one needs to avoid at most 6 colors (at most 3 colors in its neighborhood and 3 witnesses of its neighbors). Since $|X| = 7$, then the only remaining case is when $|\{\phi(b), \phi(c), \phi(d), \phi(e)\}| = 3$. In this case, $\phi(d) \in \{\phi(b), \phi(c)\}$. It is possible to recolor b in such a way that c stops being a critical vertex in ϕ. This makes $|X| \leq 6$, so there is an available color to be $\phi(a)$. Thus, ϕ is an odd 7-coloring of G, which is a contradiction. The rest follows by a careful case analysis, similar to the previous one. □

Our main result, Theorem 3, follows as a consequence of Lemma 1 and Theorem 2. However, rather than relying on Theorem 2, we prefer to derive the result using our Lemma 2, whose proof is simpler due to the imposed restriction on $\Delta(G)$.

Lemma 1 shows us that any counterexample G to Theorem 3 must be 2-connected and 4-regular. The next result shows that such a planar graph must contain one of the structures in Figure 1.



(a) Adjacent 3-faces. **(b) Intersecting 3-faces.** **(c) 3-face adjacent to a 4-face.**

Figure 1. Forbidden structures. Squigly lines represent a path of length at least 1.

Lemma 2. *If G is a 4-regular, 2-connected planar graph, then G must contain one of the subgraphs depicted in Figure 1.*

Proof sketch. This proof follows by contradiction using the discharging method. Suppose that G does not contain any of the configurations in Figure 1. Give an initial charge of $2d(v) - 6 = 2$ for all $v \in V(G)$ and $d(f) - 6$ for all $f \in F(G)$. By Euler's formula it is easy to show that $\sum_{v \in V(G)}(2d(v) - 6) + \sum_{f \in F(G)}(d(f) - 6) = -12$, so the total charge of the graph G is -12.

Now, apply the following three discharging rules:

(R1) Any vertex donates 1 of charge to every adjacent 3-face.
(R2) Any vertex donates $\frac{1}{2}$ of charge to every adjacent 4-face.
(R3) Any vertex donates $\frac{1}{5}$ of charge to every adjacent 5-face.

After the rules are applied, each element has a new charge. Note, however, that no charge was created or destroyed, so the total charge of the graph is kept the same, -12.

A 6^+-face initially has $d(f) - 6 \geq 0$ of charge and does not donate or receive charge, so it ends with at least 0 of charge (what it started with). A 5-face initially has -1 of charge and receives $\frac{1}{5}$ from each of its five vertices, so it ends with 0 of charge. A 4-face initially has -2 of charge and receives $\frac{1}{2}$ from each of its four vertices, so it ends with 0 of charge. A 3-face initially has -3 of charge and receives 1 from each of its three vertices, so it ends with 0 of charge.

The vertices initially have 2 of charge, and they can donate in several ways to their adjacent faces. Suppose a vertex a donates charge for an adjacent 3-face. Since G does not contain the configurations in Figures 1a and 1b, then a does not donate charge for any other 3-face. Since G does not contain the configuration in Figure 1c, then the other faces around a must be either a 4-face and two 5^+-faces or only 5^+-faces. Thus a ends with at least $2 - 1 - \frac{1}{2} - 2\frac{1}{5} > 0$ of charge. If a does not donate charge for 3-faces, then it donates to at most four adjacent 4-faces, also ending with at least 0 of charge.

Assuming that G contains no configurations from Figure 1, after moving the charges by applying the discharging rules, we are able to conclude that all the vertices and faces of G have a positive amount of charge, which contradicts the fact that the global charge of G is negative. Therefore, G must contain at least one of the configurations defined in Figure 1. $\square$

We are now ready to prove Theorem 3.

Proof of Theorem 3. Let G be a minimal counterexample, that is, a planar graph with $\Delta(G) \leq 4$ for which $\chi_o(G) > 7$. Lemma 1 guarantees that G is 4-regular, 2-connected, and does not contain one of the configurations defined in Figure 1. Lemma 2 guarantees that G must contain one of the configurations defined in Figure 1. Therefore, there does not exist a minimal counterexample for the theorem, and so the result follows. $\square$

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