Complexity of the Girth Problem on Arc-colored Digraphs *

Manoel Campelo[2], Rafael de Paiva Lima[1], Cláudia Linhares Sales[1]

[1] Departamento de Computação
Universidade Federal do Ceará (UFC)

[2]Departamento de Estatística e Matemática Aplicada
Universidade Federal do Ceará (UFC)

mcampelo@ufc.br, rafael.limafilho@gmail.com, linhares@dc.ufc.br

Abstract. *The girth of a graph $G = (V, E)$ is the length of its smallest cycle. This parameter is usually used as a support to obtain information about bounds or complexity of several other graph problems. In digraphs $D = (V, A)$, the girth of D is determined by the length of the smallest directed cycle. In this work, we deal with the problem of determining the girth of edge-colored graphs, where the girth is the number of colors of a cycle with the smallest number of colors. This problem is open for general graphs. In this paper we prove that its decision version is NP-complete for arc-colored digraphs. Additionally, we prove that it is W[2]-hard on this class when parameterized by the number of colors to be minimized.*

1. Introduction

An *edge-colored graph* (also known as a labeled graph) is a tuple $G = (V, E, L, \ell)$ where $G = (V, E)$ is a graph, L is a set of colors/labels, and $\ell : E \to L$ producing a coloring of $E(G)$ that is not necessarily proper. Several classical graph optimization problems have been adapted to edge-colored graphs and extensively studied, with the aim of determining their computational complexity and developing algorithms to solve them whenever possible. The optimization criteria are the adjusted to account for the labels. For example, in an edge-colored graph, the minimum path is a path with a minimum number of colors, and its girth is the number of colors of a cycle with a minimum number of colors.

It turns out that several polynomial optimization problems on general (unlabeled) graphs become NP-hard on edge-colored graphs. In fact, the minimum label spanning tree [Broersma and Li 1997, Chang and Shing-Jiuan 1997], the minimum label (s, t)-path [Broersma et al. 2005], minimum label (s, t)-cut [Zhang et al. 2011] and the maximum flow-minimum label [Granata et al. 2013] are all NP-hard on edge-colored graphs and polynomially solvable on general graphs.

In this paper, we focus on the problem of determining the girth of edge-colored graphs, introduced in [Broersma et al. 2005]. From now on, we refer to the *colored girth* as the adjusted concept for edge-colored graphs. As an example of the interest in knowing this parameter, it was observed in [Figueredo 2020, Figueredo and Campêlo 2020] that a spanning forest with at most k colors and minimum number of trees can be found in

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polynomial time if the colored girth is greater than k. In particular, the problem of determining if an edge-colored graph has a spanning tree with at most k colors is decidable in polynomial time. Although previous works in the literature claimed to establish the NP-hardness of the colored girth problem, it has been correctly pointed out that such proofs are insufficient to confirm this complexity [Figueredo 2020]. Therefore, the complexity of determining the colored girth remains an open problem.

In this work, we show that determining the girth of an arc-colored digraph is NP-hard. Moreover, our reduction also can be used to obtain information about the parameterized complexity of the problem, leading us to conclude that it is actually W[2]-hard.

2. Notation and Previous results

Let $G = (V, E, L, \ell)$ be an edge-colored graph and $D = (V, A, L, \ell)$ be an arc-colored digraph. In this paper, all graphs and digraphs are finite, without loops or parallel edges or parallel arcs. As usual, we simplify the notation, denoting by uv an edge with extremities u and v and by $\overrightarrow{uv}$ an arc with tail u and head v. Formally, the girth of G (D), denoted by $g_c(G)$ ($g_c(D)$) is the number of colors of a (directed) cycle of G (D) with a minimum number of colors. Additionally, we use $c(H)$ to denote the number of colors arising in any subgraph H of G. Let us now formally define the two central problems of this paper:

Problem GIRTH-ECOL-GRAPH Given an edge-colored graph $G = (V, E, L, \ell)$ and a positive integer k, decide if G contains a cycle C with at most k colors.

Problem GIRTH-ACOL-DIGRAPH Given an arc-colored digraph $D = (V, A, L, \ell)$ and a positive integer k, decide if D contains a directed cycle C with at most k colors.

In [Broersma et al. 2005], the authors introduced the minimum labeled s, t-path problem and proved its NP-hardness. Formally, they proved that finding a path between two given vertices s and t of an edge-colored graph G with at most k colors is NP-hard.

In the same paper, they showed a flawed proof of the NP-completeness of determining if G contains a cycle with at most k colors through a given vertex u of G, and concluded, as a consequence, that determining the girth of G would be NP-hard.

They proposed a reduction from the k-labeled s, t-path problem to the k-labeled u-cycle problem that creates a new graph G' by adding an universal vertex u with all the new edges incident to u receiving a new color 0. Then, they claimed that an s, t-path with at most k colors in G exists if and only if there exists in G' a cycle through u with at most $k + 1$ colors. That claim is false, since for any $xy \in G$, there is a 2-colored cycle u, x, y in G' regardless the existence of an $s - t$ path with at most k colors in G.

A correct reduction to the k-labeled u-cycle was presented in [Figueredo 2020], where in G' the new vertex u is adjacent only to s and t and the edges incident to u receive a new color 0. Nevertheless, it is still not true that, as a consequence of this result, the GIRTH-ECOL-GRAPH problem is NP-complete, since checking the minimum colored cycle through all the vertices of G is not proven to be the only method to compute the colored girth of G.

3. Our results

In this section, we first prove that GIRTH-ACOL-DIGRAPH Problem is NP-complete by showing a reduction from the Set Cover problem to it. Recall that it was proved by

[Karp 1972] that the latter is NP-complete. Let us formalize the Set Cover problem.

Problem SET-COVER Given a finite set $U = \{u_1, u_2, \ldots, u_n\}$ and a family of subsets $\mathcal{S} = \{S_1, S_2, \ldots, S_l\}$ where $S_i \subseteq U$ for $1 \leq i \leq \ell$ and an integer k, decide if there exists a covering $\mathcal{C} \subseteq \mathcal{S}$ such that $\bigcup_{S \in \mathcal{C}} S = U$ and $|\mathcal{C}| \leq k$.

Theorem 1. *The GIRTH-ACOL-DIGRAPH problem is NP-complete.*

Proof. It is trivial to check that this problem is in NP, as we can verify in polynomial time if a given subset of arcs C of D with at most k colors is a directed cycle.

Let us now show a polynomial time reduction from the SET-COVER problem. Given an instance $(U, \mathcal{S})$ of the SET-COVER problem, we construct an arc-colored digraph D as follows (see Figure 1):

1. We set $V(D) = \{u_1, u_2, \ldots, u_n\} \bigcup \{v_{i_j} | \forall i, j \text{ such that } u_i \in S_j\}$.
2. We define the color set as $L = \{1, 2, \ldots, \ell\}$, where each color corresponds to a subset S_i in $\mathcal{S}$.
3. For every S_j and $u_i \in S_j$, we add arcs $\overrightarrow{u_i v_{i_j}}$ and $\overrightarrow{v_{i_j} u_{(i \bmod n)+1}}$, both colored j.

We claim that there exists a set cover of size at most k if and only if there exists a directed cycle in D with at most k colors.

Let $\mathcal{C} \subseteq \mathcal{S}$ be a set cover such that $\bigcup_{S \in \mathcal{C}} S = U$ and $|\mathcal{C}| \leq k$. It means that for every $u_i \in U$, there exists $S_j \in \mathcal{C}$, such that $u_i \in S_j$. Therefore, the construction of D guarantees that for each vertex u_i, there is at least one $(u_i, u_{(i \bmod n)+1})$-path colored j, and the union of these monochromatic paths form a cycle with the same number of colors as the subsets of $\mathcal{C}$, that is, with at most k colors.

Conversely, let C be a directed cycle of D such that $|c(C)| \leq k$. Observe that any directed cycle of D contains $\{u_1 \ldots u_n\}$. Moreover, observe that, by construction, U is a stable set, and then any directed cycle contains monochromatic directed paths $(u_i, v_{i_j}, u_{(i \bmod n)+1})$, for every $u_i \in U$, which were included only if u_i belonged to some subset S_j, and in this case this monochromatic path was colored j. Therefore, a set cover $\mathcal{C}$ can be obtained by including S_j for each color $j \in c(C)$. This implies that $\bigcup_{S \in \mathcal{C}} S = U$ and $|\mathcal{C}| = |c(C)| \leq k$. □

Concerning the parameterized complexity of our problem, observe that the SET-COVER problem parameterized by k is W[2]-hard [Downey and Fellows 1998]. As a consequence, with the same reduction used in Theorem 1, we obtain that the following result:

Corollary 1. *The GIRTH-ACOL-DIGRAPH problem, with parameter k, is W[2]-hard.*

Because of Corollary 1, we can also say something about the existence of aproximative algorithms for the GIRTH-ACOL-DIGRAPH problem, since unless classes FPT (Fixed-Parameter Tractable) and W[2]-hard collapse, there is no Efficient Polynomial-Time Approximation Scheme (EPTAS) for W[2]-hard problems [Cesati and Trevisan 1997, Bazgan 1995, dos Santos and Souza 2015], we have the following additional result about the difficulty of the GIRTH-ACOL-DIGRAPH problem.

Corollary 2. *Unless FPT=W[2]-hard, the GIRTH-ACOL-DIGRAPH problem does not admit an EPTAS.*

It is also worth noting that, for any minimum label problem that is polynomially solvable in monochromatic graphs with n vertices, if $|L| = c$ is fixed, then finding a structure with at most k colors, for example, would be solved by checking all the graphs induced by k colors and seeing if they contain the desired structure. This requires $f(n) \cdot \binom{c}{k}$ time, for some polynomial function $f(n)$. Since $k \leq c$, the binomial would remain a constant factor.

We conclude this section providing an example of the reduction of Theorem 1. Consider the following instance of the SET-COVER problem:

$$U = \{u_1, u_2, u_3, u_4, u_5\},\ \mathcal{S} = \{S_1, S_2, S_3\},\ k = 2$$

$$S_1 = \{u_1, u_2, u_4\},\ S_2 = \{u_1, u_4\},\ S_3 = \{u_3, u_4, u_5\}$$

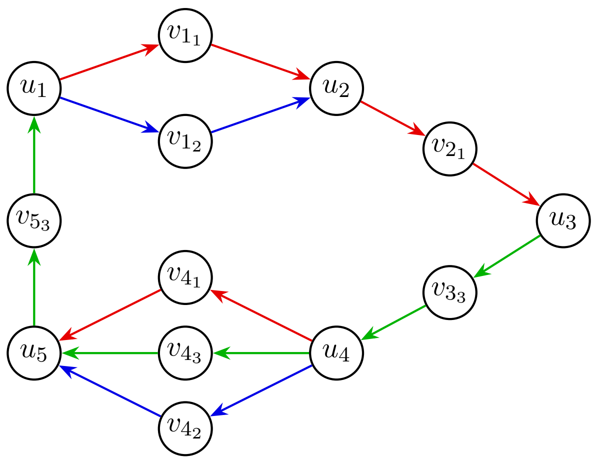

Figure 1. Edge-colored graph form the Reduction: red, blue and green are related to S_1, S_2 and S_3, respectively.

As illustrated in Figure 1, each vertex u_i can reach $u_{(i \bmod n)+1}$ via a monochromatic 2-path, and the existence of this 2-path colored j implies that $u_i \in S_j$. In our example, there is a directed cycle with colors $j = 1$ (red) and $j = 3$ (green), as well as a cover with subsets S_1 and S_3.

4. Conclusion

This work contributes another example to the class of problems that, while solvable in polynomial time on general graphs, become NP-hard when extended to edge-colored graphs. A natural open question left is determining the complexity of the GIRTH-ECOL-GRAPH problem. Observe that a trivial modification of our proof (replacing the arcs of D by edges) will eventually produce cycles with 4 edges and at most two colors, making this reduction useless for the equivalence to the SET-COVER problem.

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