Distance-k Domination Number in Triangular Matchstick Graphs and Triangulated Rectangular Grids

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Abstract. *A distance-k dominating set of a graph is a set of vertices such that every vertex lies within distance k of some vertex in the set; its minimum size is the distance-k domination number γ_k. We study γ_k for triangular matchstick graphs and triangulated rectangular grids. For triangulated rectangular grids, we give upper bounds for γ and γ_2. For triangular matchstick graphs T_d, we correct two upper-bound statements of Harris et al. [Harris et al. 2020], improve the bound for $k = 1$, derive an upper bound for $k = 2$ and relate γ_k to the radius of T_d.*

1. Introduction

We consider simple connected graphs and use standard terminology [Bondy and Murty 2008, Diestel 2017]. For vertices u and v, let $dist(u, v)$ denote their distance. A set $D \subseteq V$ is a *distance-k dominating set* if every vertex is at distance at most k from some vertex of D. The minimum size of such a set is the *distance-k domination number* $\gamma_k(G)$. For $k = 1$, $\gamma_1(G)$ is the classical domination number $\gamma(G)$, a parameter studied extensively in the literature [Slater 1976, Haynes et al. 2020].

Domination and distance-k domination have been extensively studied in planar and grid-like graphs. Matheson and Tarjan conjectured that sufficiently large triangulated planar graphs satisfy $\gamma(G) \leq n/4$. In particular, this bound holds for triangular matchstick graphs [Hongo and Campos 2013]. Several additional results concerning this conjecture have been established [Matheson and Tarjan 1996, King and Pelsmajer 2010, Plummer and Zha 2016, Špacapan 2020, Christiansen et al. 2024].

Domination in grid graphs has been widely studied, and such graphs often serve as a reference point for related grid-like graph families. Exact values of the domination number for rectangular grids $G_{p\times q}$ of fixed width p were first obtained for $p \leq 4$ in [Jacobson and Kinch 1983] and later extended by [Chang and Clark 1993] for $p = 5, 6$. Exact values for all sufficiently large grids were determined by [Gonçalves et al. 2011]. General upper bounds for distance-k domination in rectangular grids have also been established [Fata et al. 2013, Blessing et al. 2015]. These results motivate the study of distance-k domination in triangulated rectangular grids.

In this work, we study γ_k for two classes of graphs: *triangular matchstick graphs* and *triangulated rectangular grids*. Our main contributions are:

- We establish upper bounds for γ and γ_2 in several families of triangulated rectangular grids; the bounds for γ are tight, and the bounds for γ_2 are tight for widths $p = 2$ and $p = 3$ (Theorems 1 and 2).

- We identify a flaw in the tiling-based upper-bound arguments of Harris et al. [Harris et al. 2020] and give corrected bounds for $k = 1$ and $k = 2$.
- We present an improved bound on γ for triangular matchstick graphs (Theorem 3).
- We establish an upper bound on γ_2 for triangular matchstick graphs (Theorem 4).
- We determine the radius of triangular matchstick graphs and relate it to γ_k (Theorem 5).

2. Triangulated Rectangular Grids

Given paths P_p and P_q, the grid $G_{p\times q}$ is the Cartesian product $P_p \square P_q$. Let $\mathcal{G}_{p\times q}$ denote the graphs obtained by triangulating all inner faces of $G_{p\times q}$. The vertices of $G_{p\times q}$ are labeled v_{ij} with $0 \leq i < p$ and $0 \leq j < q$. Figures 1a and 1b illustrate $G_{2\times 3}$ and a graph in $\mathcal{G}_{2\times 3}$, respectively.

(a) $G_{2\times 3}$. **(b) An instance of** $\mathcal{G}_{2\times 3}$.

Figure 1. A grid (a) and a triangulated rectangular grid (b).

Jacobson and Kinch [Jacobson and Kinch 1983] determined exact values of the domination number for grids of small fixed widths. They showed that, when $p \leq q$ and $p \in \{2, 3, 4\}$,

$$\gamma(G_{p\times q}) = \begin{cases} \left\lceil \dfrac{q+1}{2} \right\rceil, & \text{if } p = 2; \\ \left\lceil \dfrac{3q+1}{4} \right\rceil, & \text{if } p = 3; \\ q + 1, & \text{if } p = 4 \text{ and } q \in \{5, 6, 9\}; \\ q, & \text{if } p = 4 \text{ and } q \notin \{5, 6, 9\}. \end{cases}$$

Motivated by these results, we establish upper bounds on $\gamma(G)$ for graphs in $\mathcal{G}_{p\times q}$ when $p \in \{2, 3, 4\}$, showing that triangulating the faces reduces the domination number compared to the corresponding grid.

Theorem 1. *Let $G \in \mathcal{G}_{p\times q}$ with $p \in \{2, 3, 4\}$ and $q \geq p$. Then*

$$\gamma(G) \leq \begin{cases} \left\lfloor \dfrac{q+1}{2} \right\rfloor, & \textit{if } p = 2, \\ \left\lfloor \dfrac{2q+2}{3} \right\rfloor, & \textit{if } p = 3, \\ q, & \textit{if } p = 4. \end{cases}$$

Moreover, each bound is tight.

Fata et al. established general upper bounds for distance-k domination in grids [Fata et al. 2013, Theorem V.9]. They showed that, when $p \leq q$ and $k \geq 2$,

$$\gamma_k(G_{p\times q}) \leq \left\lceil \frac{(p+2k)(q+2k)}{2k^2+2k+1} + \frac{2k^2+2k+1}{4} \right\rceil .$$

In particular, for $k = 2$, this specializes to

$$\gamma_2(G_{p\times q}) \leq \left\lceil \frac{(p+4)(q+4)}{13} + \frac{13}{4} \right\rceil .$$

Theorem 2. *Let $G \in \mathcal{G}_{p\times q}$ with $p \in \{2,3,4\}$ and $q \geq p$. Then*

$$\gamma_2(G) \leq \begin{cases} \dfrac{q}{4}, & \textit{if } p = 2 \textit{ and } q \equiv 0 \pmod 4, \\ \left\lfloor \dfrac{q}{4} \right\rfloor + 1, & \textit{if } p = 2 \textit{ and } q \not\equiv 0 \pmod 4, \\ \dfrac{q}{3}, & \textit{if } p = 3 \textit{ and } q \equiv 0 \pmod 3, \\ \left\lfloor \dfrac{q}{3} \right\rfloor + 1, & \textit{if } p = 3 \textit{ and } q \not\equiv 0 \pmod 3, \\ \dfrac{q}{2}, & \textit{if } p = 4 \textit{ and } q \equiv 0 \pmod 2, \\ \left\lfloor \dfrac{q}{2} \right\rfloor + 1, & \textit{if } p = 4 \textit{ and } q \not\equiv 0 \pmod 2. \end{cases}$$

Moreover, the bounds are tight for $p = 2$ and $p = 3$.

In several cases, triangulating the internal faces strictly reduces the domination number compared to that of the corresponding grid. For graphs in $\mathcal{G}_{2\times q}$, this occurs when $q \equiv 0 \pmod 2$; for $\mathcal{G}_{3\times q}$, when $q \geq 6$; and for $\mathcal{G}_{4\times q}$, when $q \in \{5, 6, 9\}$.

For distance-2 domination, the upper bounds obtained for graphs in $\mathcal{G}_{p\times q}$ with $p \in \{2,3,4\}$ are strictly smaller than the general bounds for grids established by [Fata et al. 2013].

3. Triangular Matchstick Graph

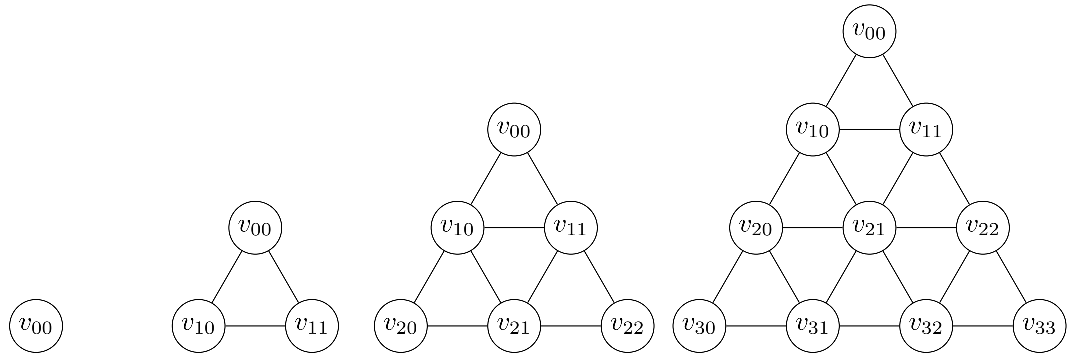
The *triangular lattice* T_∞ is the infinite graph whose vertex set is

$$(x, y) = \left(a - \frac{b}{2}, \frac{b\sqrt{3}}{2} \right), \quad a, b \in \mathbb{Z},$$

where two vertices are adjacent if their distance is 1. A *triangular matchstick graph* T_d is the subgraph of T_∞ induced by vertices with $0 \leq b \leq a \leq d$. Figure 2 illustrates T_d for $0 \leq d \leq 3$. Harris et al. derived upper bounds for the distance-k domination number of triangular matchstick graphs using an efficient broadcast pattern on the triangular lattice.

Theorem ([Harris et al. 2020], Theorem 4.2)**.** If $d = 7q + \beta$, with $0 \leq \beta \leq 6$, then

$$\gamma(T_d) \leq \begin{cases} 3q(q+2), & \text{if } \beta = 0, \\ 3(q+1)(q+3), & \text{if } \beta \neq 0. \end{cases}$$



(a) T_0. **(b)** T_1. **(c)** T_2. **(d)** T_3.

Figure 2. T_d **for** $0 \leq d \leq 3$.

Theorem ([Harris et al. 2020], Theorem 4.3)**.** If $d = 19q + \beta$, with $0 \leq \beta \leq 18$, then

$$\gamma_2(T_d) \leq \begin{cases} 9q(q+1), & \text{if } \beta = 0, \\ 9(q+1)(q+2), & \text{if } \beta \neq 0. \end{cases}$$

However, these upper bounds do not hold. The proofs of Theorems 4.2 and 4.3 of Harris et al. [Harris et al. 2020] overlook vertices of degree 6 at the intersections of the tiles in the corresponding decompositions, leading to incorrect double-counting adjustments. This is further evidenced by Lemma 3.1 of Harris et al. [Harris et al. 2020], whose lower bound asymptotically exceeds those upper bounds for large d. We next give corrected upper-bound statements for the constructions underlying those theorems, using the adjustment described above.

For $k = 1$, if $d = 7q + \beta$, with $0 \leq \beta \leq 6$, then

$$\gamma(T_d) \leq \begin{cases} 3.5q^2 + 4.5q + 1, & \text{if } \beta = 0, \\ 3.5q^2 + 11.5q + 9, & \text{if } \beta \neq 0. \end{cases}$$

Moreover, we can improve this bound as follows.

Theorem 3. *If $d = 7q + \beta$, where $q \geq 1$ and $0 \leq \beta \leq 6$, then*

$$\gamma(T_d) \leq 3.5q^2 + (\beta + 4.5)q + \beta - 1.$$

For $k = 2$ we obtain the following result.

Theorem 4. *If $d = 19q + \beta$, with $0 \leq \beta \leq 18$, then*

$$\gamma_2(T_d) \leq \begin{cases} 9.5q^2 + 7.5q + 1, & \text{if } \beta = 0, \\ 9.5q^2 + 26.5q + 18, & \text{if } \beta \neq 0. \end{cases}$$

Finally, we show that when k is sufficiently large with respect to the radius of the graph, all vertices can be dominated by a single vertex.

Theorem 5. *For every $d \geq 0$, the radius of T_d is $\left\lceil \frac{2d}{3} \right\rceil$.*

Corollary 1. *Let $d \geq 0$. Then $\gamma_k(T_d) = 1$ for every $k \geq \left\lceil \frac{2d}{3} \right\rceil$.*

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Appendix

4. Proofs for Triangulated Rectangular Grids

4.1. Proof of Theorem 1

Recall that we want to prove that

$$\gamma(G) \leq \begin{cases} \left\lfloor \dfrac{q+1}{2} \right\rfloor, & \text{if } p = 2, \\ \left\lfloor \dfrac{2q+2}{3} \right\rfloor, & \text{if } p = 3, \\ q, & \text{if } p = 4, \end{cases} \quad \text{for } G \in \mathcal{G}_{p\times q} \text{ and } q \geq p,$$

and that this bound is tight.

Proof. We consider separately the cases $p = 2$ and $p \in \{3, 4\}$. The first requires a direct periodic construction along the two rows, while the latter consist of partitioning G into consecutive blocks of fixed width, each of which admits a dominating set of the same size, together with an additional set that dominates the bounded remainder near the boundary.

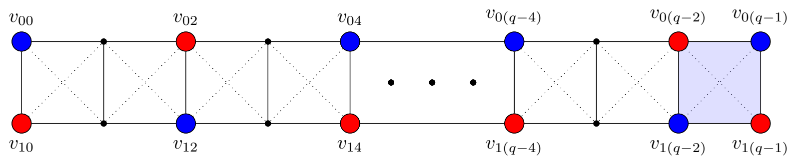
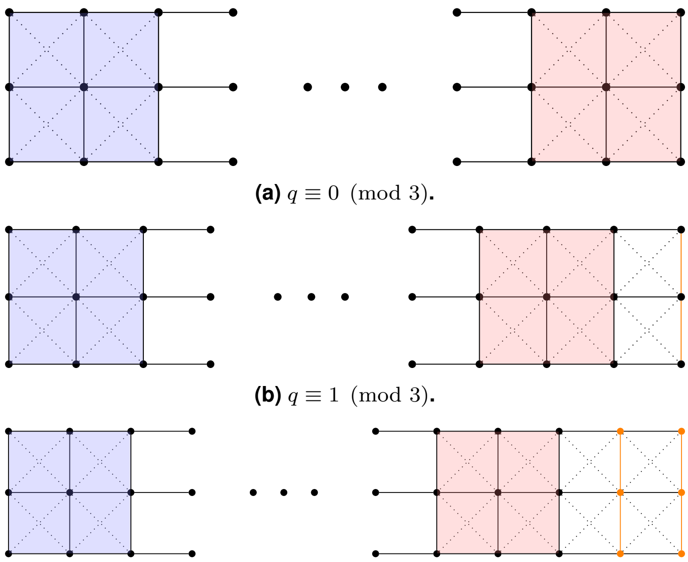


Figure 3. Instance of $\mathcal{G}_{2\times q}$. Blue vertices are in D and red vertices are in D'.

Case $p = 2$. For even q, consider the periodic construction shown in Figure 3, which se-



(a) $q \equiv 0 \pmod 3$.

(b) $q \equiv 1 \pmod 3$.

(c) $q \equiv 2 \pmod 3$.

Figure 4. Instance of $\mathcal{G}_{3\times q}$ partitioned into subgraphs isomorphic to graphs in $\mathcal{G}_{3\times 3}$.

lects vertices alternately on the two rows every two columns, together with one additional vertex at the right boundary. This produces two candidate dominating sets, D and D', each of size $\left\lfloor \frac{q+1}{2} \right\rfloor + 1$. Depending on the orientation of the final diagonal edge, we select the candidate set that contains both of its endpoints; in this case, one boundary vertex becomes redundant, and removing it yields a dominating set of size $\left\lfloor \frac{q+1}{2} \right\rfloor$. For odd q, it is known for grids that

$$\gamma(G_{2\times q}) = \left\lceil \frac{q+1}{2} \right\rceil$$

[Jacobson and Kinch 1983], which equals $\left\lfloor \frac{q+1}{2} \right\rfloor$ since q is odd. As G is obtained from $G_{2\times q}$ by adding edges (which cannot increase the domination number), it follows that

$$\gamma(G) \leq \gamma(G_{2\times q}) = \left\lfloor \frac{q+1}{2} \right\rfloor.$$

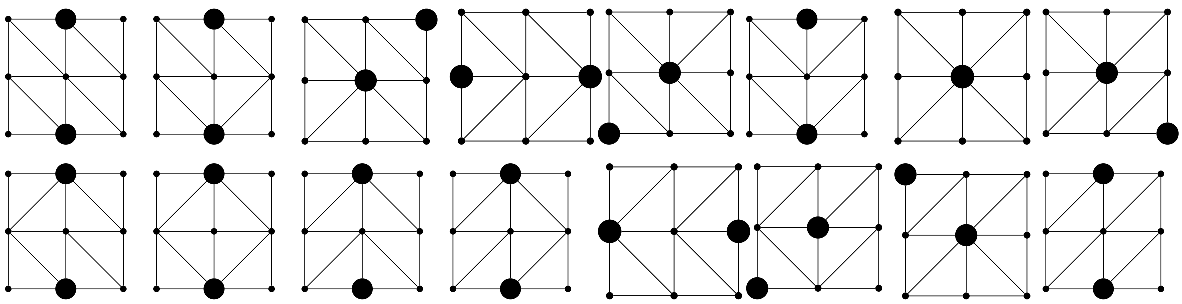


Figure 5. Dominating sets for all instances of $\mathcal{G}_{3\times 3}$.

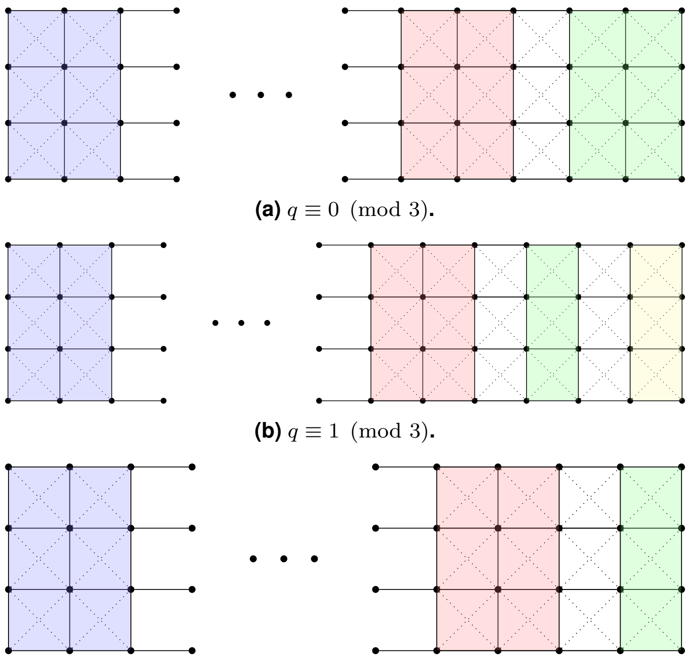
Case $p \in \{3, 4\}$. The graph G can be partitioned into consecutive blocks of fixed width together with a bounded remainder at the right boundary.

For $p = 3$, the graph G can be partitioned into consecutive blocks of width 3, each inducing a subgraph isomorphic to a graph in $\mathcal{G}_{3\times 3}$, as illustrated in Figure 4. Each such block admits a dominating set of size at most 2 (see Figure 5). This partition is completed at the right boundary by adding one or two paths isomorphic to P_3, each of which can be dominated by one vertex. Consequently,

$$\gamma(G) \leq \begin{cases} \frac{2q}{3}, & q \equiv 0 \pmod 3, \\ \frac{2(q-1)}{3} + 1, & q \equiv 1 \pmod 3, \\ \frac{2(q-2)}{3} + 2, & q \equiv 2 \pmod 3, \end{cases} = \left\lfloor \frac{2q+2}{3} \right\rfloor.$$

For $p = 4$, we partition G into consecutive blocks isomorphic to graphs in $\mathcal{G}_{3\times 4}$, as illustrated in Figure 6, each of which can be dominated by 3 vertices. This partition is completed at the right boundary by adding one or two subgraphs isomorphic to graphs in $\mathcal{G}_{2\times 4}$, as in Figure 6; each such subgraph can be dominated by 2 vertices. Consequently,

$$\gamma(G) \leq \begin{cases} \frac{q}{3} \cdot 3, & q \equiv 0 \pmod 3, \\ \frac{q-4}{3} \cdot 3 + 2 \cdot 2, & q \equiv 1 \pmod 3, \\ \frac{q-2}{3} \cdot 3 + 2, & q \equiv 2 \pmod 3, \end{cases} = q.$$

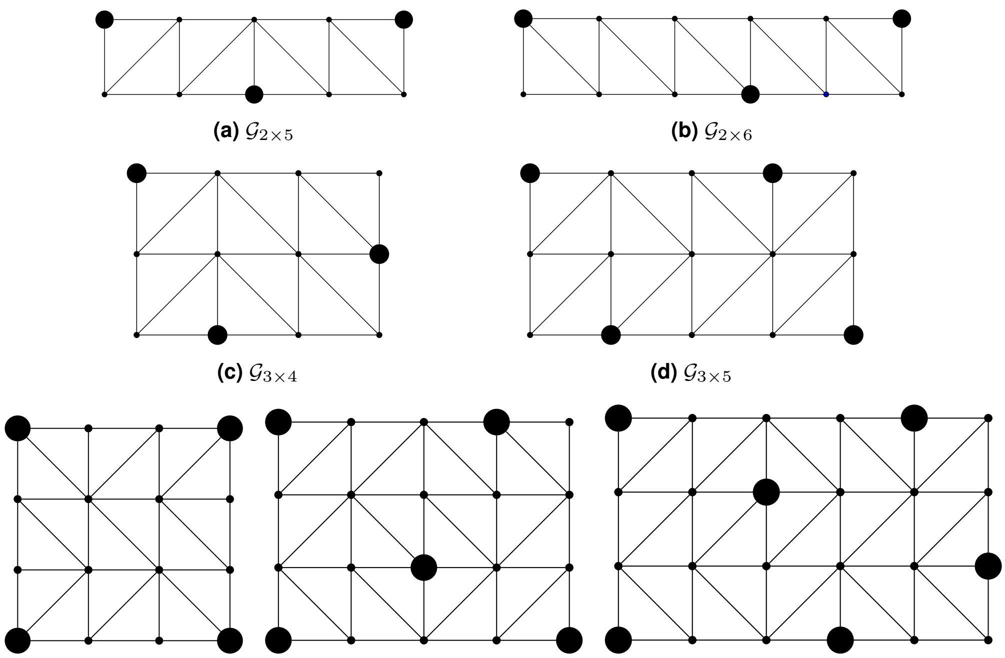


(a) $q \equiv 0 \pmod 3$.

(b) $q \equiv 1 \pmod 3$.

(c) $q \equiv 2 \pmod 3$.

Figure 6. Instance of $\mathcal{G}_{4\times q}$ partitioned into subgraphs isomorphic to graphs in $\mathcal{G}_{3\times 4}$.



(a) $\mathcal{G}_{2\times 5}$ **(b)** $\mathcal{G}_{2\times 6}$

(c) $\mathcal{G}_{3\times 4}$ **(d)** $\mathcal{G}_{3\times 5}$

(e) $\mathcal{G}_{4\times 4}$ **(f)** $\mathcal{G}_{4\times 5}$ **(g)** $\mathcal{G}_{4\times 6}$

Figure 7. Minimum dominating sets (black vertices) for small instances of $\mathcal{G}_{p\times q}$ with $p \in \{2, 3, 4\}$ where the upper bounds are tight.

The bounds are tight for the smallest instances of each family, as illustrated in Figures 5 and 7. In all cases, the black vertices form a minimum dominating set, since their closed neighborhoods are pairwise disjoint.

□

4.2. Proof of Theorem 2

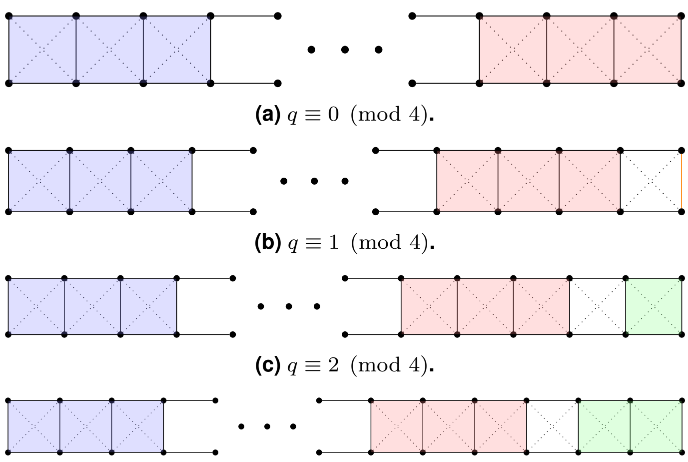
Recall that we want to prove that

$$\gamma_2(G) \leq \begin{cases} \frac{q}{4}, & \text{if } p = 2 \text{ and } q \equiv 0 \pmod 4, \\ \left\lfloor \frac{q}{4} \right\rfloor + 1, & \text{if } p = 2 \text{ and } q \not\equiv 0 \pmod 4, \\ \frac{q}{3}, & \text{if } p = 3 \text{ and } q \equiv 0 \pmod 3, \\ \left\lfloor \frac{q}{3} \right\rfloor + 1, & \text{if } p = 3 \text{ and } q \not\equiv 0 \pmod 3, \\ \frac{q}{2}, & \text{if } p = 4 \text{ and } q \equiv 0 \pmod 2, \\ \left\lfloor \frac{q}{2} \right\rfloor + 1, & \text{if } p = 4 \text{ and } q \not\equiv 0 \pmod 2. \end{cases}$$

Moreover, the bounds are tight for $p = 2$ and $p = 3$.

Proof. In each case, we partition G into consecutive blocks of fixed width that are distance-2 dominated by one vertex; if q is not a multiple of the block width, the remaining subgraph at the right boundary is distance-2 dominated by one additional vertex.

Case $p = 2$.



(a) $q \equiv 0 \pmod 4$.

(b) $q \equiv 1 \pmod 4$.

(c) $q \equiv 2 \pmod 4$.

(d) $q \equiv 3 \pmod 4$.

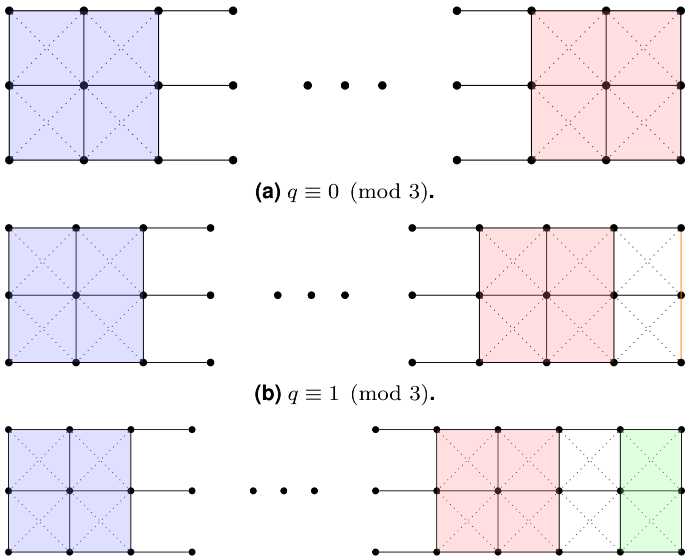
Figure 8. Instance of $\mathcal{G}_{2\times q}$ partitioned into subgraphs isomorphic to graphs in $\mathcal{G}_{2\times 4}$.

First, it is straightforward to verify that every instance of $\mathcal{G}_{2\times 2}$ and $\mathcal{G}_{2\times 3}$ is distance-2 dominated by one vertex. Moreover, every instance of $\mathcal{G}_{2\times 4}$ is distance-2 dominated by one vertex: if $G \in \mathcal{G}_{2\times 4}$, then either $v_{01}v_{12} \in E$ or $v_{11}v_{02} \in E$, and hence $\{v_{01}\}$ or $\{v_{11}\}$ is a distance-2 dominating set. If $q \equiv 0 \pmod 4$, then G can be partitioned into

$\frac{q}{4}$ subgraphs isomorphic to graphs in $\mathcal{G}_{2\times 4}$ (Figure 8a), yielding $\gamma_2(G) \leq \frac{q}{4}$. Otherwise, G can be partitioned into $\lfloor \frac{q}{4} \rfloor$ subgraphs isomorphic to graphs in $\mathcal{G}_{2\times 4}$ and one additional subgraph isomorphic to P_2, to a graph in $\mathcal{G}_{2\times 2}$, or to a graph in $\mathcal{G}_{2\times 3}$ (Figures 8b–8d), which is distance-2 dominated by one vertex. Consequently,

$$\gamma_2(G) \leq \begin{cases} \dfrac{q}{4}, & q \equiv 0 \pmod 4, \\ \left\lfloor \dfrac{q}{4} \right\rfloor + 1, & q \not\equiv 0 \pmod 4. \end{cases}$$

Case $p = 3$.



(a) $q \equiv 0 \pmod 3$.

(b) $q \equiv 1 \pmod 3$.

(c) $q \equiv 2 \pmod 3$.

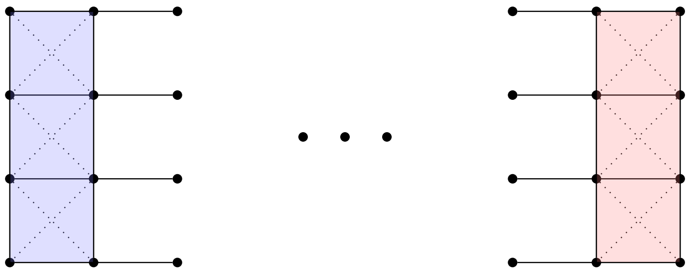
Figure 9. Instance of $\mathcal{G}_{3\times q}$ partitioned into subgraphs isomorphic to graphs in $\mathcal{G}_{3\times 3}$.

Every instance of $\mathcal{G}_{3\times 3}$ is distance-2 dominated by one vertex, namely $\{v_{11}\}$. If $q \equiv 0 \pmod 3$, then G can be partitioned into $\frac{q}{3}$ subgraphs isomorphic to graphs in $\mathcal{G}_{3\times 3}$ (Figure 9a), yielding $\gamma_2(G) \leq \frac{q}{3}$. Otherwise, G can be partitioned into $\lfloor \frac{q}{3} \rfloor$ such subgraphs and one additional subgraph isomorphic to P_3 or to a graph in $\mathcal{G}_{2\times 3}$ (Figures 9b–9c); the former is distance-2 dominated by one vertex, and the latter is distance-2 dominated by one vertex by the previously established case $p = 2$. Consequently,

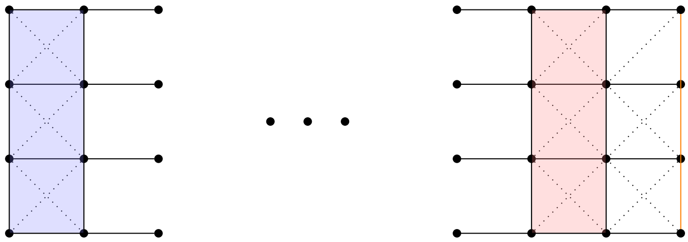
$$\gamma_2(G) \leq \begin{cases} \dfrac{q}{3}, & q \equiv 0 \pmod 3, \\ \left\lfloor \dfrac{q}{3} \right\rfloor + 1, & q \not\equiv 0 \pmod 3. \end{cases}$$

Case $p = 4$.

By the case $p = 2$, every instance of $\mathcal{G}_{2\times 4}$ is distance-2 dominated by one vertex. If $q \equiv 0 \pmod 2$, then G can be partitioned into $\frac{q}{2}$ subgraphs isomorphic to graphs in $\mathcal{G}_{2\times 4}$ (Figure 10a), yielding $\gamma_2(G) \leq \frac{q}{2}$. Otherwise, G can be partitioned into $\lfloor \frac{q}{2} \rfloor$ such subgraphs and one subgraph isomorphic to P_4 (Figure 10b); since P_4 is distance-2



(a) $q \equiv 0 \pmod 2$.



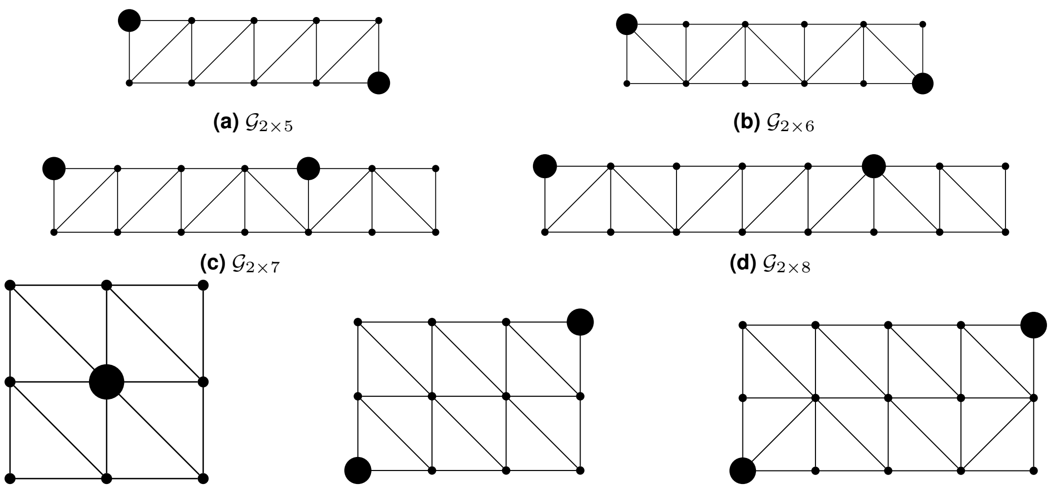
(b) $q \equiv 1 \pmod 2$.

Figure 10. Instance of $\mathcal{G}_{4\times q}$ partitioned into subgraphs isomorphic to graphs in $\mathcal{G}_{2\times 4}$.

dominated by one vertex, we obtain

$$\gamma_2(G) \le \begin{cases} \dfrac{q}{2}, & q \equiv 0 \pmod 2, \\ \left\lfloor \dfrac{q}{2} \right\rfloor + 1, & q \not\equiv 0 \pmod 2. \end{cases}$$

The bounds are tight for the smallest instances in the cases $p = 2$ and $p = 3$, as illustrated in Figure 11. In this figure, the black vertices form a minimum distance-2 dominating set, since the sets of vertices at distance at most 2 from each selected vertex are pairwise disjoint.



(a) $\mathcal{G}_{2\times 5}$ **(b)** $\mathcal{G}_{2\times 6}$

(c) $\mathcal{G}_{2\times 7}$ **(d)** $\mathcal{G}_{2\times 8}$

(e) $\mathcal{G}_{3\times 3}$ **(f)** $\mathcal{G}_{3\times 4}$ **(g)** $\mathcal{G}_{3\times 5}$

Figure 11. Minimum distance-2 dominating sets (black vertices) for small instances of $\mathcal{G}_{p\times q}$ with $p \in \{2, 3\}$ where the upper bounds are tight.

□

4.3. Corollary for Distance Domination ($k = 1$) in Triangulated Rectangular Grids

Corollary 2. *Let $G_{p\times q}$ be a grid and let $G \in \mathcal{G}_{p\times q}$ with $p \in \{2,3,4\}$ and $q \geq p$.*

$$\gamma(G) < \gamma(G_{p\times q}) \qquad \textit{if} \qquad \begin{cases} p = 2 \textit{ and } q \equiv 0 \pmod 2, \\ \textit{or } p = 3 \textit{ and } q \geq 6, \\ \textit{or } p = 4 \textit{ and } q \in \{5,6,9\}. \end{cases}$$

Proof. For $G_{p\times q}$, the domination numbers are known [Jacobson and Kinch 1983]. By Theorem 1, the corresponding triangulated instances satisfy

$$\gamma(G) \leq \begin{cases} \left\lfloor \dfrac{q+1}{2} \right\rfloor, & p = 2, \\ \left\lfloor \dfrac{2q+2}{3} \right\rfloor, & p = 3, \\ q, & p = 4. \end{cases}$$

Case $p = 2$**.** It is known that $\gamma(G_{2\times q}) = \lceil \frac{q+1}{2} \rceil$. If q is even, then

$$\left\lfloor \frac{q+1}{2} \right\rfloor < \left\lceil \frac{q+1}{2} \right\rceil,$$

and therefore $\gamma(G) < \gamma(G_{2\times q})$.

Case $p = 3$**.** It is known that $\gamma(G_{3\times q}) = \lceil \frac{3q+1}{4} \rceil$. For $q \geq 6$,

$$\frac{2q+2}{3} < \frac{3q+1}{4},$$

which implies

$$\left\lfloor \frac{2q+2}{3} \right\rfloor < \left\lceil \frac{3q+1}{4} \right\rceil,$$

and hence $\gamma(G) < \gamma(G_{3\times q})$.

Case $p = 4$**.** For $q \in \{5,6,9\}$, it is known that $\gamma(G_{4\times q}) = q + 1$. Since $\gamma(G) \leq q$, we obtain

$$\gamma(G) < \gamma(G_{4\times q}).$$

□

4.4. Observation for Distance-2 Domination in Triangulated Rectangular Grids

Observation 1. Let $G_{p\times q}$ be a grid and let $G \in \mathcal{G}_{p\times q}$ with $p \in \{2,3,4\}$ and $q \geq p$. Then the upper bound for $\gamma_2(G)$ given by Theorem 2 is strictly smaller than the upper bound for $\gamma_2(G_{p\times q})$ obtained in [Fata et al. 2013].

Proof. Upper bounds for the distance-2 domination number of grids $G_{p\times q}$ are given in [Fata et al. 2013]. By Theorem 2, the triangulated instances satisfy the following bounds, which are valid for all cases of the theorem:

$$\gamma_2(G) \le \begin{cases} \left\lfloor \frac{q}{4} \right\rfloor + 1, & p = 2, \\ \left\lfloor \frac{q}{3} \right\rfloor + 1, & p = 3, \\ \left\lfloor \frac{q}{2} \right\rfloor + 1, & p = 4. \end{cases}$$

Case $p = 2$. It is known that

$$\gamma_2(G_{2\times q}) \le \left\lceil \frac{6(q+4)}{13} + \frac{13}{4} \right\rceil .$$

Since

$$\left\lfloor \frac{q}{4} \right\rfloor + 1 \le \frac{q}{4} + 1 < \frac{6(q+4)}{13} + \frac{13}{4} \quad \text{for } q \ge 2,$$

we obtain that the upper bound for $\gamma_2(G)$ is strictly smaller than the upper bound for $\gamma_2(G_{2\times q})$.

Case $p = 3$. It is known that

$$\gamma_2(G_{3\times q}) \le \left\lceil \frac{7(q+4)}{13} + \frac{13}{4} \right\rceil .$$

Since

$$\left\lfloor \frac{q}{3} \right\rfloor + 1 \le \frac{q}{3} + 1 < \frac{7(q+4)}{13} + \frac{13}{4} \quad \text{for } q \ge 3,$$

we obtain that the upper bound for $\gamma_2(G)$ is strictly smaller than the upper bound for $\gamma_2(G_{3\times q})$.

Case $p = 4$. It is known that

$$\gamma_2(G_{4\times q}) \le \left\lceil \frac{8(q+4)}{13} + \frac{13}{4} \right\rceil .$$

Since

$$\left\lfloor \frac{q}{2} \right\rfloor + 1 \le \frac{q}{2} + 1 < \frac{8(q+4)}{13} + \frac{13}{4} \quad \text{for } q \ge 4,$$

we obtain that the upper bound for $\gamma_2(G)$ is strictly smaller than the upper bound for $\gamma_2(G_{4\times q})$. □

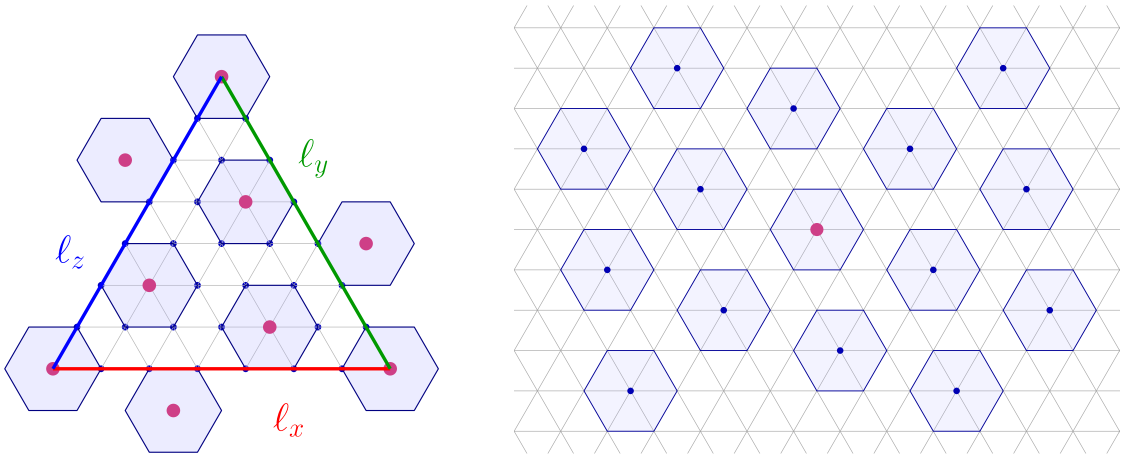
5. Proofs for Triangular Matchstick Graphs

5.1. Discussion of the argument of Harris et al.

In this subsection, we demonstrate that the arguments in the proofs of [Harris et al. 2020, Theorems 4.2 and 4.3] are incorrect. We also provide corrected statements together with valid proofs.

We denote by ℓ_x, ℓ_y, and ℓ_z the three sides of the outer triangular boundary of T_d, as illustrated in Figure 12a. Specifically, ℓ_x denotes the horizontal base of T_d, while ℓ_y and ℓ_z denote the right and left boundary sides, respectively.

Harris et al. based their constructions on an *efficient pattern* $P_{t,r}$ for (t, r)-broadcast domination on the infinite triangular grid T_∞ [Harris et al. 2020, Theorem 4.1]. In particular, for $t = 2$ and $r = 1$ this yields the periodic pattern $P_{2,1}$, which we use as the template underlying their tiling-based upper-bound arguments. Figure 12b depicts $P_{2,1}$ on T_∞, with a pattern vertex at the origin $(0, 0)$.



(a) Side notation and a dominating set for T_7.

(b) Pattern $P_{2,1}$ on T_∞, with a pattern vertex at the origin $(0, 0)$.

Figure 12. Side notation, the pattern $P_{2,1}$, and its application to T_7.

When $r = 1$, $(t, 1)$-broadcast domination coincides with distance-$(t - 1)$ domination. In particular, the pattern $P_{2,1}$ corresponds to classical domination ($k = 1$), and $P_{3,1}$ corresponds to distance-2 domination ($k = 2$). We begin by analyzing each of [Harris et al. 2020, Theorems 4.2 and 4.3] in the case $\beta = 0$, showing that the stated bounds do not hold in this setting.

In [Harris et al. 2020, Theorem 4.2], the authors cover T_d with copies of T_7 using the efficient pattern $P_{2,1}$ and argue as follows:

- The vertex at the intersection of the sides ℓ_x and ℓ_z of T_d is placed at $(0, 0)$.
- The efficient pattern $P_{2,1}$ contains the vertex at $(0, 0)$.
- If a vertex belongs to the pattern, then all vertices at distance 7 from it along the directions of the grid also belong to the pattern.
- The graph T_d can be covered with q^2 copies of T_7.
- Each copy of T_7 is dominated by at least 9 vertices of the pattern (Figure 12a).

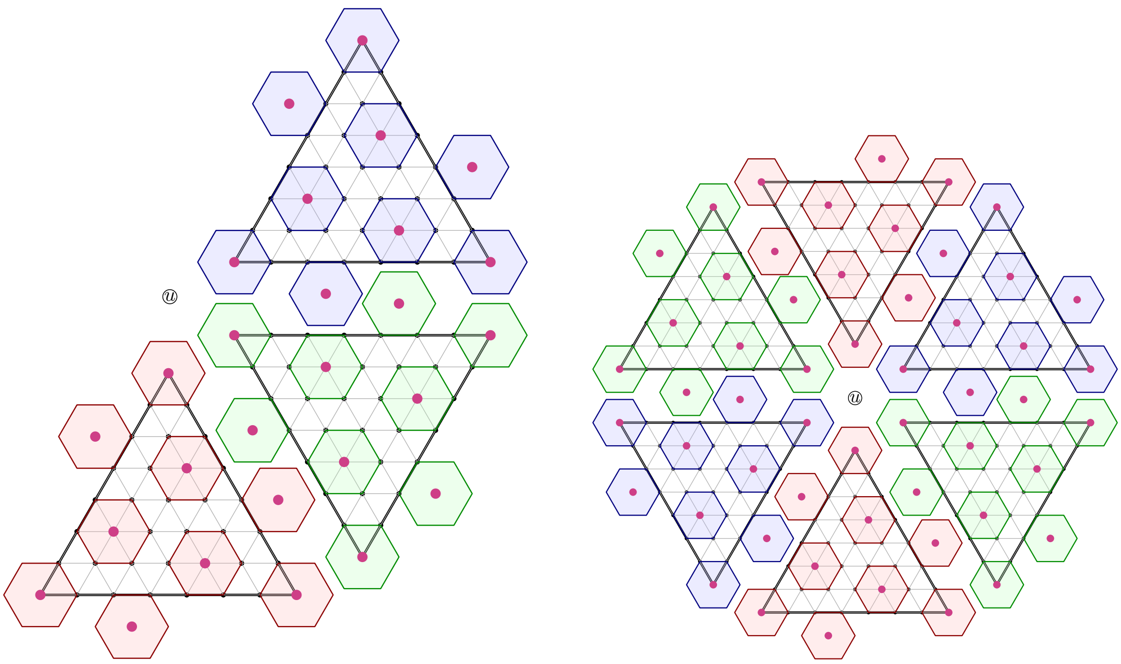
From this, they obtain the preliminary bound

$$\gamma(T_d) \leq 9q^2.$$

Since vertices on shared internal edges of adjacent copies of T_7 are counted twice, Harris et al. subtract a correction term for this double counting. They argue that this correction equals $6q(q - 1)$, since each shared side between two copies of T_7 contains four duplicated dominant vertices. The number of such internal sides in the decomposition of T_d is $\frac{3q(q-1)}{2}$, multiplying by 4 yields $6q(q-1)$, leading to

$$\gamma(T_d) \leq 3q(q+2).$$

This correction is valid for vertices lying on at most two internal edges of the decomposition, but it fails at vertices incident with six copies in the tiling. Figure 13a shows a degree-4 vertex u lying on two internal edges: in this situation, u is counted three times and subtracted twice, leaving one occurrence, as intended. In contrast, Figure 13b shows a degree-6 vertex u, which lies on six internal edges. Applying the same rule would subtract u once for each incident internal edge, removing all occurrences of u from the count. Consequently, u (and the vertices it is supposed to dominate) are no longer accounted for, and the resulting total is underestimated.



(a) A vertex of degree 4. **(b) A vertex of degree 6.**

Figure 13. Application of the method to vertices of different degrees.

The smallest instance where this issue appears is T_{21}. Harris et al. claim that $\gamma(T_{21}) \leq 45$. However, when the same construction is carried out consistently, the pattern $P_{2,1}$ induces a dominating set of size 46 for T_{21} (Figure 14).

The same issue appears in the proof of Theorem 4.3 of [Harris et al. 2020], where T_d is covered by copies of T_{19}. Vertices of degree 6 again occur at points where six copies meet, producing the same invalid cancellation. The first failure arises in T_{57}, which contains such a configuration.

Finally, we obtain both an asymptotic and a concrete contradiction between [Harris et al. 2020, Lemma 3.1] and [Harris et al. 2020, Theorems 4.2 and 4.3]. For $k = 1$ and $d = 7q$, Lemma 3.1 yields

$$\gamma(T_d) \;\geq\; \left\lceil \frac{(d+2)(d+1)}{14} \right\rceil = 3.5q^2 + O(q),$$

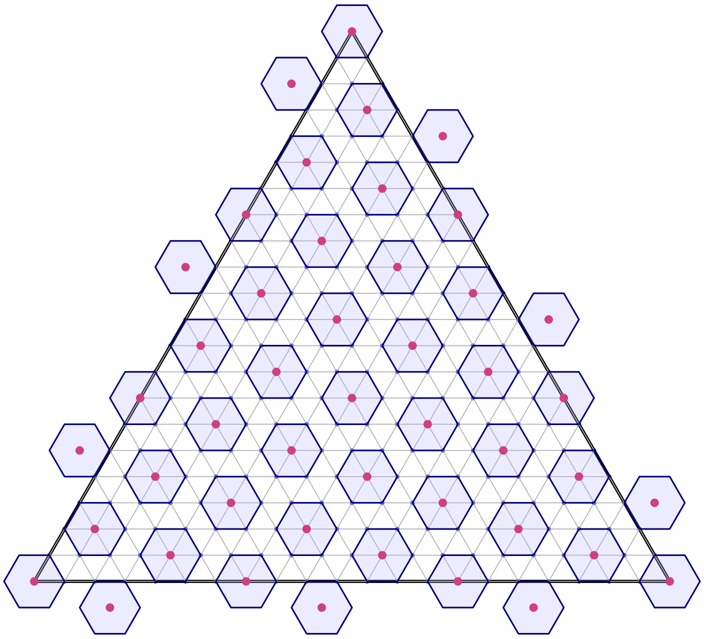


Figure 14. Dominating set for T_{21} using 46 vertices of $P_{2,1}$.

whereas Theorem 4.2 asserts

$$\gamma(T_{7q}) \leq 3q(q+2) = 3q^2 + O(q).$$

Hence, for sufficiently large d, the lower bound must exceed the proposed upper bound. This already occurs at $d = 63$ (i.e., $q = 9$), where

$$\left\lceil \frac{65 \cdot 64}{14} \right\rceil = 298 \quad \text{but} \quad 3 \cdot 9 \cdot 11 = 297,$$

so the claimed upper bound is strictly smaller than a valid lower bound, and the formula in Theorem 4.2 cannot be correct.

Similarly, for $k = 2$ and $d = 19q$, Lemma 3.1 gives

$$\gamma_2(T_d) \;\geq\; \left\lceil \frac{(d+2)(d+1)}{38} \right\rceil = 9.5q^2 + O(q),$$

whereas Theorem 4.3 asserts

$$\gamma_2(T_{19q}) \leq 9q(q+1) = 9q^2 + O(q).$$

Again, for sufficiently large d the lower bound exceeds the proposed upper bound, and this already occurs at $d = 285$ (i.e., $q = 15$), where

$$\left\lceil \frac{287 \cdot 286}{38} \right\rceil = 2161 \quad \text{but} \quad 9 \cdot 15 \cdot 16 = 2160.$$

Therefore, the formula in Theorem 4.3 cannot be correct either.

5.2. Correction to the upper bound in Theorem 4.2 of Harris et al.

In this subsection, we show that if $d = 7q + \beta$, with $0 \leq \beta \leq 6$, then

$$\gamma(T_d) \leq \begin{cases} 3.5q^2 + 4.5q + 1, & \text{if } \beta = 0, \\ 3.5q^2 + 11.5q + 9, & \text{if } \beta \neq 0, \end{cases}$$

adjusting the upper bound stated by Harris et al. [Harris et al. 2020].

Lemma 1. *Let $d = 7q$, with $q \geq 1$. Consider the tiling of T_d by copies of T_7 induced by the pattern $P_{2,1}$. Then the number of interior vertices of T_d that have degree 6, belong to the pattern $P_{2,1}$ and lie at the intersection of six copies of T_7 is*

$$I_q = \frac{(q-1)(q-2)}{2}.$$

Proof of Lemma 1. We proceed by induction on q. For $q = 1$, the graph T_7 has no interior vertex of degree 6 satisfying the stated conditions, hence

$$I_1 = 0 = \frac{(1-1)(1-2)}{2}.$$

Assume that for some $q > 1$ we have

$$I_{q-1} = \frac{(q-2)(q-3)}{2}.$$

The graph T_{7q} is obtained from $T_{7(q-1)}$ by adding one row of copies of T_7 along the side ℓ_x, as illustrated in Figure 15.

Observe that the added row makes exactly $q - 2$ vertices on the side ℓ_x of $T_{7(q-1)}$—highlighted in blue in Figure 15 and belonging to $P_{2,1}$—change from degree 4 in $T_{7(q-1)}$ to degree 6 in T_{7q}, as they become interior vertices lying at the intersection of six copies of T_7. Therefore,

$$\begin{aligned} I_q &= I_{q-1} + (q-2) \\ &= \frac{(q-2)(q-3)}{2} + (q-2) \\ &= \frac{(q-1)(q-2)}{2}. \end{aligned}$$

Consequently, $I_q = \frac{(q-1)(q-2)}{2}$ for all $q \geq 1$. □

Lemma 2. *If $\beta = 0$, then $\gamma(T_d) \leq 3.5q^2 + 4.5q + 1$.*

Proof of Lemma 2. If $\beta = 0$, T_d can be decomposed into q^2 copies of T_7, each dominated by 9 vertices of the pattern $P_{2,1}$ (see Figure 12a). Hence $\gamma(T_d) \leq 9q^2$.

Following the argument of Harris et al. [Harris et al. 2020], removing the double counting along shared sides subtracts $6q(q-1)$ vertices of $P_{2,1}$. This subtraction also

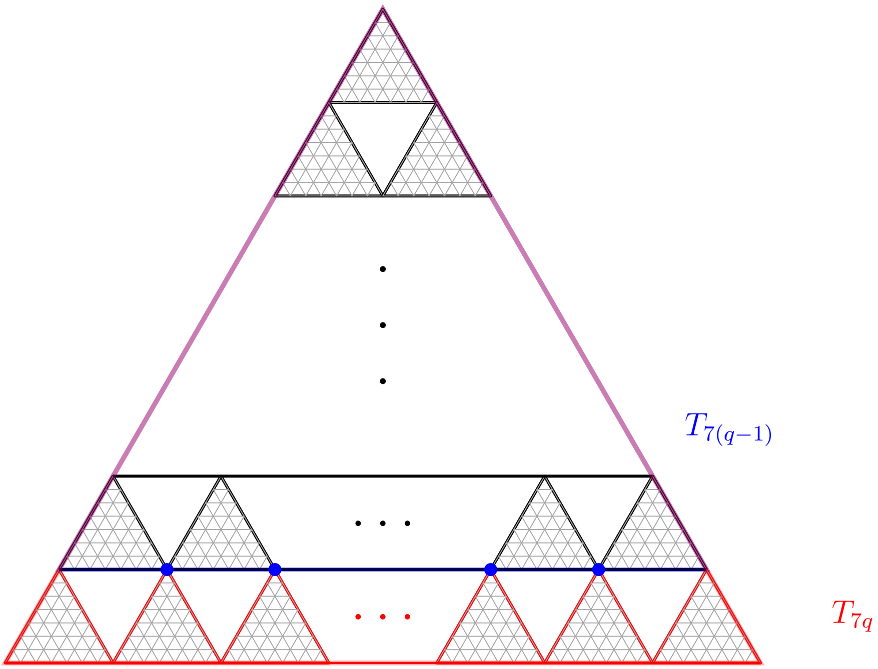


Figure 15. Addition of a row of copies of T_7 to $T_{7(q-1)}$ to obtain T_{7q} and the vertices that change from degree 4 to degree 6.

removes the interior degree-6 vertices of $P_{2,1}$ that must be counted once, so these I_q vertices are added back. Therefore,

$$\begin{aligned}\gamma(T_d) &\leq 9q^2 - 6q(q-1) + I_q \\ &= 9q^2 - 6q^2 + 6q + \frac{q^2 - 3q + 2}{2} \\ &= \frac{7q^2 + 9q + 2}{2} \\ &= 3.5q^2 + 4.5q + 1.\end{aligned}$$

This finishes the proof of Lemma 2. □

Lemma 2 establishes the corrected upper bound in the case $\beta = 0$. If $d = 7q + \beta$ with $1 \leq \beta \leq 6$, then T_d is a subgraph of $T_{7(q+1)}$, and hence

$$\gamma(T_d) \leq \gamma\big(T_{7(q+1)}\big)\,.$$

Applying Lemma 2 with q replaced by $q + 1$ yields

$$\gamma(T_d) \leq 3.5(q+1)^2 + 4.5(q+1) + 1 = 3.5q^2 + 11.5q + 9.$$

Consequently, if $d = 7q + \beta$ with $0 \leq \beta \leq 6$, then

$$\gamma(T_d) \leq \begin{cases} 3.5q^2 + 4.5q + 1, & \text{if } \beta = 0, \\ 3.5q^2 + 11.5q + 9, & \text{if } \beta \neq 0, \end{cases}$$

which completes the correction of the bounds for all $0 \leq \beta \leq 6$.

5.3. Proof of Theorem 3

Recall that we want to prove that

$$\gamma(T_d) \leq 3.5q^2 + (\beta + 4.5)q + \beta - 1,$$

where $d = 7q + \beta$, $q \geq 1$ and $0 \leq \beta \leq 6$.

The proof is carried out in three phases that apply to all values of β. In Phase I, we extend a dominating set of T_{7q} to dominate $T_{7q+\beta}$ and obtain an initial upper bound for $\gamma(T_d)$. In Phase II, we translate the graph within the infinite triangular grid while preserving the initial distribution of the vertices of $P_{2,1}$, so that the upper bound on $\gamma(T_d)$ is maintained and may decrease in some cases. In Phase III, we perform local optimizations that further decrease the size of the dominating set. The final bounds follow by combining the contributions of these three phases for each value of β.

Phase I: Construction of $T_{7q+\beta}$ from T_{7q}. Recall that $d = 7q + \beta$. The graph T_d contains a unique subgraph isomorphic to $T_{7(q-1)}$ that shares the vertex at the intersection of the sides ℓ_x and ℓ_z of T_d. Along the side ℓ_y of this subgraph, the vertices of the pattern $P_{2,1}$ occur periodically with a period of 7, including both endpoints of the side; consequently, there are exactly q such vertices on this side.

Each of these vertices lies at the intersection of the sides ℓ_x and ℓ_z of a copy of $T_{7+\beta}$ attached to $T_{7(q-1)}$, and the vertices of $P_{2,1}$ that dominate each such copy form the same pattern depending only on β. Adjacent copies may share some dominating vertices, so the total number of additional vertices depends on β and grows linearly with q.

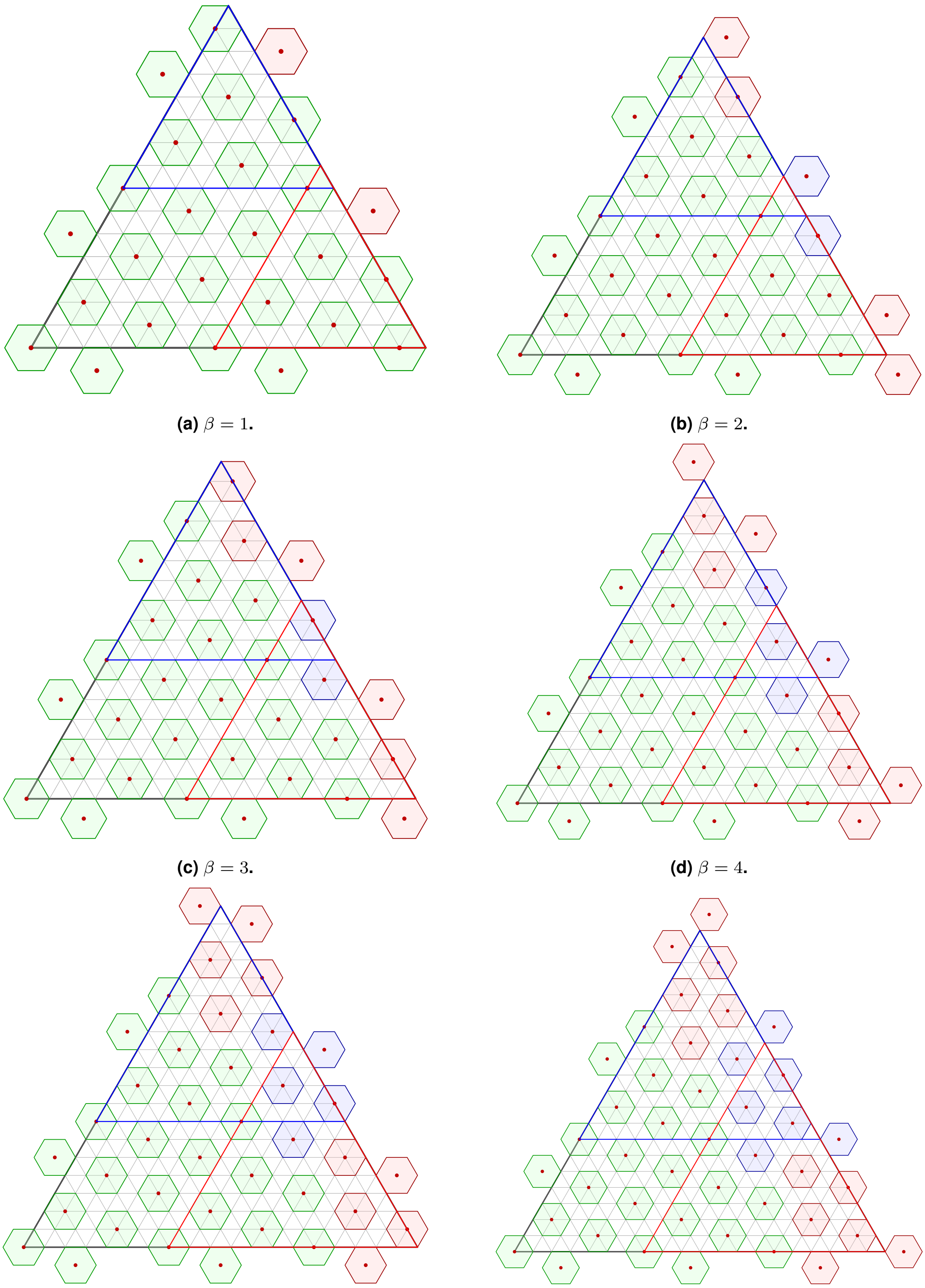
This construction yields an initial upper bound on $\gamma(T_d)$ obtained from $\gamma(T_{7q})$ by adding the vertices required to dominate the attached copies of $T_{7+\beta}$. For $\beta \geq 1$ these vertices are indicated in Figure 16, where the green vertices correspond to those already present in a dominating set of T_{7q}, while the red and blue vertices belong to the added set, with the blue vertices being those shared by two adjacent copies of $T_{7+\beta}$. For $\beta = 0$, the bound follows directly from Lemma 2. Altogether, this construction yields the following initial estimate:

$$\gamma(T_d) \leq 3.5q^2 + (\beta + 4.5)q + (\beta - 1) + a_\beta,$$

where

$$a_\beta = \begin{cases} 1, & \beta \in \{1, 3\}, \\ 2, & \beta \in \{0, 2, 4, 5\}, \\ 3, & \beta = 6. \end{cases}$$

Phase II: Translation. To optimize the number of dominating vertices, we translate the graph T_d within the infinite triangular grid T_∞ while preserving the distribution of the pattern $P_{2,1}$. A *translation* consists of shifting the graph rigidly without altering the relative positions of the vertices of $P_{2,1}$; hence the resulting set remains a valid dominating set. In particular, we apply a translation in the direction parallel to the side ℓ_z, as illustrated in Figure 17.



(a) $\beta = 1$.

(b) $\beta = 2$.

(c) $\beta = 3$.

(d) $\beta = 4$.

(e) $\beta = 5$.

(f) $\beta = 6$.

Figure 16. Phase I (construction). Extension of a dominating set from T_{7q} to $T_{7q+\beta}$ for $q = 2$ and $1 \leq \beta \leq 6$. The vertices with green hexagons form a dominating set of T_{7q}, while the red and blue vertices represent the additional vertices required to dominate the attached copies of $T_{7+\beta}$; the blue vertices are shared by two adjacent copies.

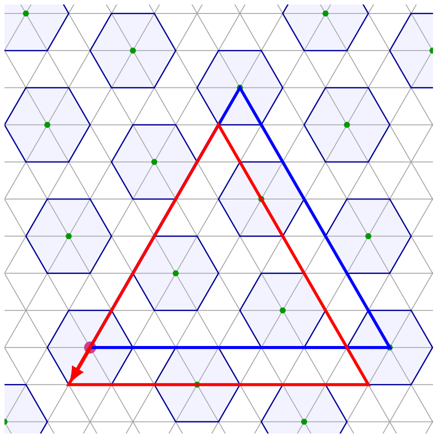


Figure 17. Translation of the graph T_7 (from blue to red) on T_∞.

Under this translation, some dominating vertices required in Phase I are no longer needed, while others must be added along the new boundary. In Figure 18, the red vertices indicate those added after the translation, and the blue vertices indicate those that are no longer required. This produces a revised upper bound on $\gamma(T_d)$ obtained after the translation, leading to the following estimate:

$$\gamma(T_d) \leq 3.5q^2 + (\beta + 4.5)q + b_\beta,$$

where

$$b_\beta = \begin{cases} 1, & \beta \in \{0,1\}, \\ 2, & \beta = 2, \\ 3, & \beta \in \{3,4\}, \\ 6, & \beta = 5, \\ 7, & \beta = 6. \end{cases}$$

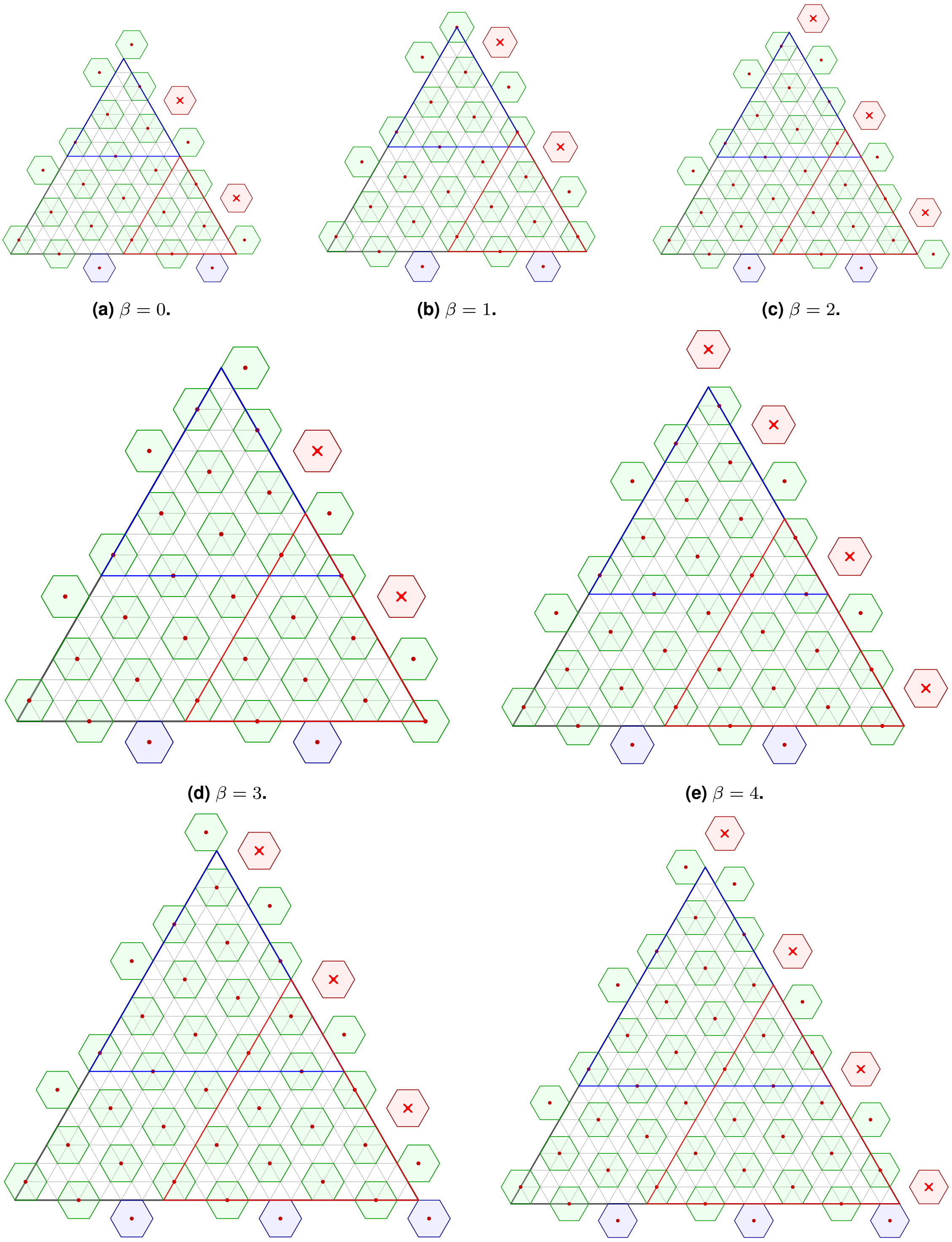
Phase III: Local optimization. In $T_{7q+\beta}$, denote by $T^U_{7q+\beta}$ the unique subgraph isomorphic to $T_{7+\beta}$ that contains the vertex at the intersection of the sides ℓ_y and ℓ_z of $T_{7q+\beta}$, and by $T^L_{7q+\beta}$ the unique subgraph isomorphic to $T_{7+\beta}$ that contains the vertex at the intersection of the sides ℓ_x and ℓ_y of $T_{7q+\beta}$.

In the final phase, we perform local optimizations inside one or both of the subgraphs $T^U_{7q+\beta}$ and $T^L_{7q+\beta}$ obtained from the translation. These optimizations consist of replacing certain subsets of dominating vertices of $P_{2,1}$ with smaller subsets that still dominate all vertices of T_d. More precisely, groups of x dominating vertices are replaced by groups of y vertices with $y < x$, without losing the domination property. The replacements depend only on β and are preserved as q increases.

The vertices removed and those added are indicated in Figure 19. This yields the final estimate

$$\gamma(T_d) \leq 3.5q^2 + (\beta + 4.5)q + \beta - 1.$$

This completes the proof of Theorem 3.

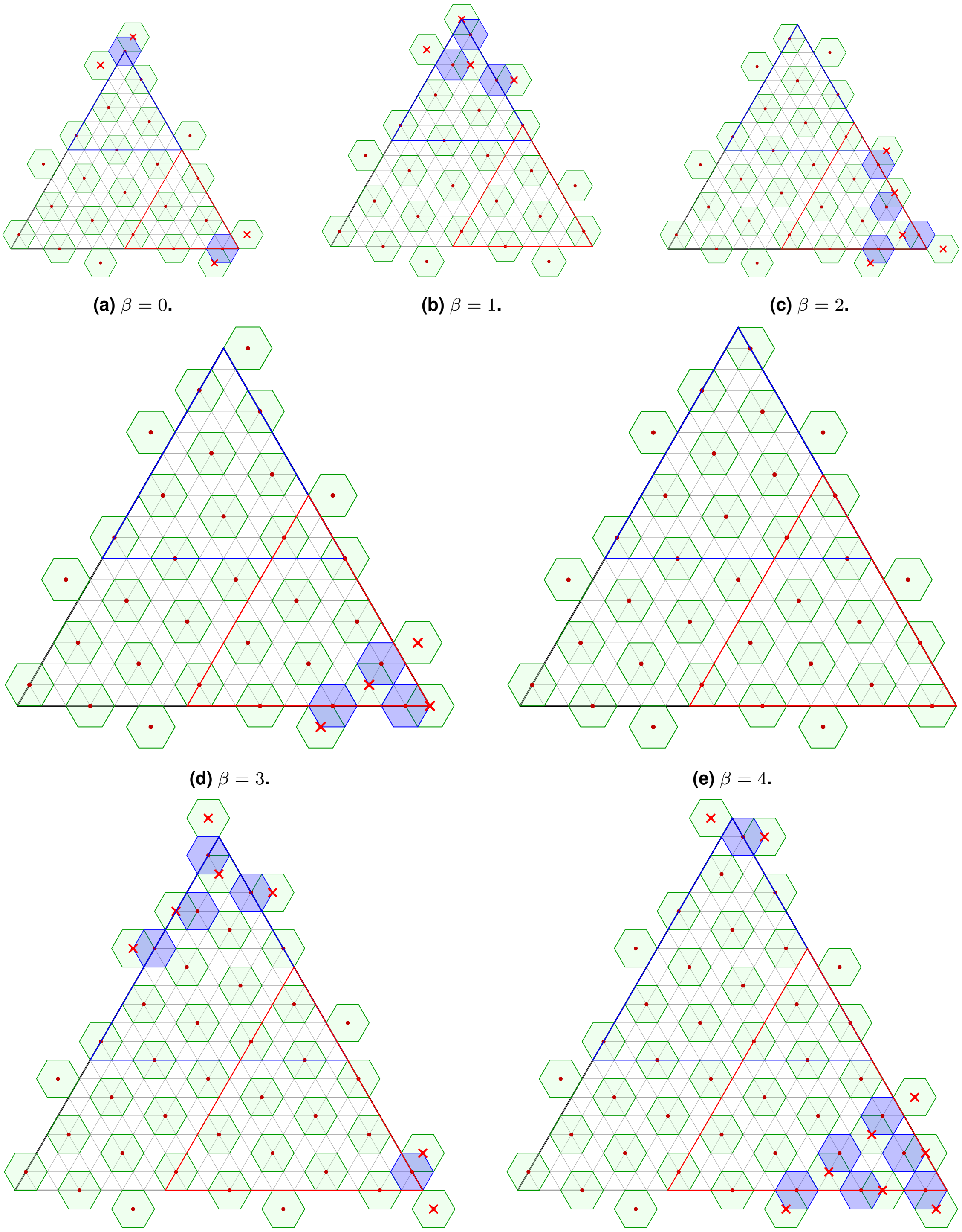


(a) $\beta = 0$. **(b)** $\beta = 1$. **(c)** $\beta = 2$.

(d) $\beta = 3$. **(e)** $\beta = 4$.

(f) $\beta = 5$. **(g)** $\beta = 6$.

Figure 18. Phase II (translation) applied to $T_{7q+\beta}$ for $q = 2$ and $0 \leq \beta \leq 6$. All graphs are translated in the same direction within the infinite triangular grid while preserving the pattern $P_{2,1}$. The red vertices indicate dominating vertices that must be added after the translation, whereas the blue vertices indicate those that are no longer required.



(a) $\beta = 0$. **(b)** $\beta = 1$. **(c)** $\beta = 2$. **(d)** $\beta = 3$. **(e)** $\beta = 4$. **(f)** $\beta = 5$. **(g)** $\beta = 6$.

Figure 19. Phase III (local optimization) applied to the translated graphs $T_{7q+\beta}$ for $0 \leq \beta \leq 6$. Certain dominating vertices of $P_{2,1}$ are replaced by smaller equivalent sets that still dominate the graph. The vertices marked with $\times$ are removed from the dominating set, while the shaded regions indicate the locations where the replacements occur.

5.4. Proof of Theorem 4

Recall that we want to prove that

$$\gamma_2(T_d) \le \begin{cases} 9.5q^2 + 7.5q + 1, & \text{if } \beta = 0, \\ 9.5q^2 + 26.5q + 18, & \text{if } \beta \neq 0, \end{cases}$$

where $d = 19q + \beta$ and $0 \le \beta \le 18$.

Lemma 3. *Let $d = 19q$, with $q \ge 1$. Consider the tiling of T_d by copies of T_{19} induced by the pattern $P_{3,1}$. Then the number of interior vertices of T_d that have degree 6, belong to the pattern $P_{3,1}$, and lie at the intersection of six copies of T_{19} is*

$$I_q = \frac{(q-1)(q-2)}{2}.$$

Proof of Lemma 3. We proceed by induction on q. For $q = 1$, the graph T_{19} has no interior vertex of degree 6 satisfying the stated conditions, hence

$$I_1 = 0 = \frac{(1-1)(1-2)}{2}.$$

Assume that for some $q > 1$ we have

$$I_{q-1} = \frac{(q-2)(q-3)}{2}.$$

The graph T_{19q} is obtained from $T_{19(q-1)}$ by adding one row of copies of T_{19} along the side ℓ_x.

Observe that the added row makes exactly $q - 2$ vertices on the side ℓ_x of $T_{19(q-1)}$—belonging to $P_{3,1}$—change from degree 4 in $T_{19(q-1)}$ to degree 6 in T_{19q}, as they become interior vertices lying at the intersection of six copies of T_{19}. Therefore,

$$\begin{aligned} I_q &= I_{q-1} + (q-2) \\ &= \frac{(q-2)(q-3)}{2} + (q-2) \\ &= \frac{(q-1)(q-2)}{2}. \end{aligned}$$

Consequently, $I_q = \frac{(q-1)(q-2)}{2}$ for all $q \ge 1$. □

By [Harris et al. 2020, Theorem 4.3], if $d = 19q + \beta$ with $0 \le \beta \le 18$, then

$$\gamma_2(T_d) \le \begin{cases} 9q(q+1), & \text{if } \beta = 0, \\ 9(q+1)(q+2), & \text{if } \beta \neq 0. \end{cases}$$

As established in our correction to the proof of Theorem 4.3, the overlap-subtraction step in [Harris et al. 2020] cancels the interior degree-6 vertices of $P_{3,1}$ occurring at points where six copies of T_{19} meet. These vertices must be counted once; hence they must be added back.

By Lemma 3, the number of such vertices is

$$I_q = \frac{(q-1)(q-2)}{2}.$$

Therefore, adding I_q to the bound of [Harris et al. 2020], Theorem 4.3 yields the corrected estimates.

Case $\beta = 0$. If $d = 19q$, then

$$\begin{aligned}
\gamma_2(T_d) &\leq 9q(q+1) + I_q \\
&= 9q^2 + 9q + \frac{(q-1)(q-2)}{2} \\
&= 9q^2 + 9q + \frac{q^2 - 3q + 2}{2} \\
&= \frac{19q^2 + 15q + 2}{2} \\
&= 9.5q^2 + 7.5q + 1.
\end{aligned}$$

Case $\beta \neq 0$. If $d = 19q + \beta$ with $1 \leq \beta \leq 18$, then T_d is a subgraph of $T_{19(q+1)}$, and therefore the bound obtained for the case $\beta = 0$ applies with q replaced by $q + 1$. Hence,

$$\begin{aligned}
\gamma_2(T_d) &\leq 9(q+1)(q+2) + I_{q+1} \\
&= 9(q^2 + 3q + 2) + \frac{q(q-1)}{2} \\
&= 9q^2 + 27q + 18 + \frac{q^2 - q}{2} \\
&= \frac{19q^2 + 53q + 36}{2} \\
&= 9.5q^2 + 26.5q + 18.
\end{aligned}$$

This proves Theorem 4.

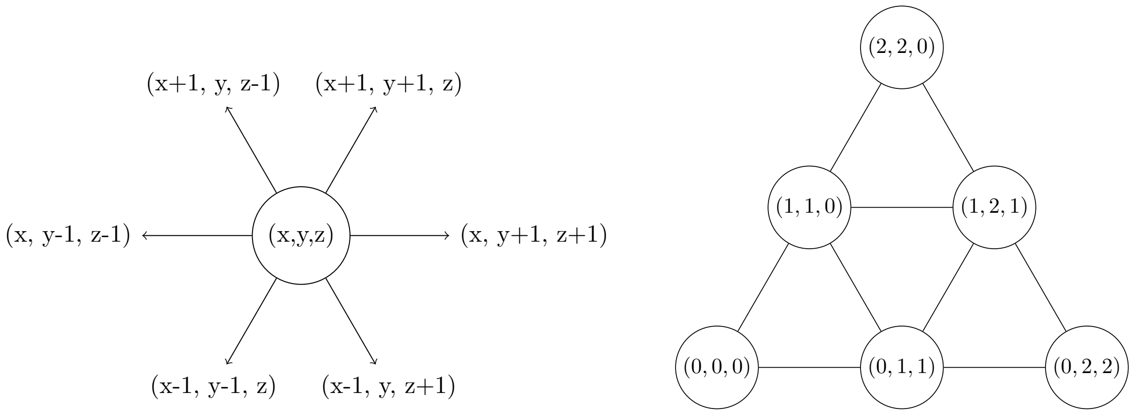
5.5. Proof of Theorem 5

Recall that we want to prove that the radius r_d of T_d equals $\lceil \frac{2d}{3} \rceil$. For this, it suffices to show that the following recurrence holds:

$$r_d = \begin{cases} d & \text{if } d < 3, \\ r_{d-3} + 2 & \text{if } d \geq 3. \end{cases}$$

We use coordinates (x, y, z) for vertices of T_d satisfying $0 \leq x, y, z \leq d$ and $y = x + z$, which uniquely identify each vertex [Bose et al. 2020]. Adjacent vertices differ by ± 1 in exactly two coordinates and coincide in the third; consequently, each vertex has at most six neighbors arranged along three principal directions (Figure 20a). The corner vertices are $(0, 0, 0)$, $(d, d, 0)$, and $(0, d, d)$, and each pair is at distance d.

The three sides of T_d are the shortest paths between the corner vertices $(0, 0, 0)$, $(d, d, 0)$, and $(0, d, d)$, denoted by ℓ_x, ℓ_y, and ℓ_z. We call $(0, 0, 0)$ the *origin*, and the side joining $(d, d, 0)$ and $(0, d, d)$ the *opposite side* (Figure 20b).



(a) Coordinates of the neighbors of a vertex (x, y, z) of T_d.

(b) Coordinates of the vertices of T_2.

Figure 20. Coordinate system in T_d and the example of T_2.

Lemma 4. *If $u = (x_u, y_u, z_u)$ and $v = (x_v, y_v, z_v)$ are vertices of T_d, then*

$$dist(u, v) = \frac{|x_u - x_v| + |y_u - y_v| + |z_u - z_v|}{2}.$$

Proof. Let $P = u_1, \ldots, u_k$ be a u–v path with $u_1 = u$ and $u_k = v$, and define

$$\phi_{ij} = |x_{u_j} - x_{u_i}| + |y_{u_j} - y_{u_i}| + |z_{u_j} - z_{u_i}|.$$

Since adjacent vertices differ by ± 1 in two coordinates and coincide in the third, $\phi_{i(i+1)} = 2$. By the triangle inequality,

$$\phi_{1k} \leq \sum_{i=1}^{k-1} \phi_{i(i+1)} = 2|P|,$$

so every u–v path has length at least $\frac{\phi_{1k}}{2}$.

For the reverse inequality, if $u \neq v$ there exists a neighbor u' of u such that

$$|x_{u'} - x_v| + |y_{u'} - y_v| + |z_{u'} - z_v| = |x_u - x_v| + |y_u - y_v| + |z_u - z_v| - 2.$$

Iterating this construction yields a u–v path of length $\frac{\phi_{1k}}{2}$. □

Lemma 5. *Let u be a vertex of $T_d = (V, E)$, and let v' be a corner vertex whose distance from u is maximal. Then $ecc(u) = dist(u, v')$.*

Proof. Let $f(x, y, z) = |x_u - x| + |y_u - y| + |z_u - z|$. If v is not a corner vertex, then v has a neighbor w such that $f(w) \geq f(v)$. Since T_d is finite, iterating this argument produces a corner vertex v' with $f(v') \geq f(v)$. □

Recall that we want to show the next recurrence for r_d:

$$r_d = \begin{cases} d & \text{if } d < 3, \\ r_{d-3} + 2 & \text{if } d \geq 3. \end{cases}$$

We proceed by induction on d. The cases $d \leq 2$ follow by inspection. Assume $d \geq 3$. Figure 21 illustrates the construction of T_d from T_{d-3} by adding three outer strips, one parallel to each side. Let c_1, c_2, c_3 be the corner vertices of T_d and c'_1, c'_2, c'_3 the corresponding corners of T_{d-3}. For $v \in T_{d-3}$ we prove $ecc(v) = ecc'(v) + 2$, where ecc' denotes eccentricity in T_{d-3}.

By Lemma 5, it suffices to show $dist(v, c_i) = dist(v, c'_i) + 2$. We verify this for $c_1 = (0, 0, 0)$ and $c'_1 = (1, 2, 1)$; the other cases are analogous. By Lemma 4, $dist(v, c_1) = \frac{|v_x|+|v_y|+|v_z|}{2}$ and $dist(v, c'_1) = \frac{|v_x-1|+|v_y-2|+|v_z-1|}{2}$. Since v belongs to the embedded copy of T_{d-3} in T_d, we have $v_x, v_z \geq 1$ and $v_y = v_x + v_z \geq 2$. Hence $|v_x - 1| = v_x - 1$, $|v_y - 2| = v_y - 2$, and $|v_z - 1| = v_z - 1$, so $dist(v, c'_1) = dist(v, c_1) - 2$ and therefore $ecc(v) = ecc'(v) + 2$. Moreover, Figure 21 shows that every vertex of T_d outside T_{d-3} lies in one of the three outer strips added around T_{d-3}, and therefore is farther from the opposite corner than some vertex of T_{d-3}, so such vertices cannot be central.

Let u be a central vertex of T_{d-3} with $ecc'(u) = r_{d-3}$. Then $ecc(u) = r_{d-3}+2$, and since the eccentricities of the vertices in the embedded copy of T_{d-3} increase by 2 when passing from T_{d-3} to T_d, the vertex u remains central. Hence $r_d = r_{d-3} + 2$, completing the proof.

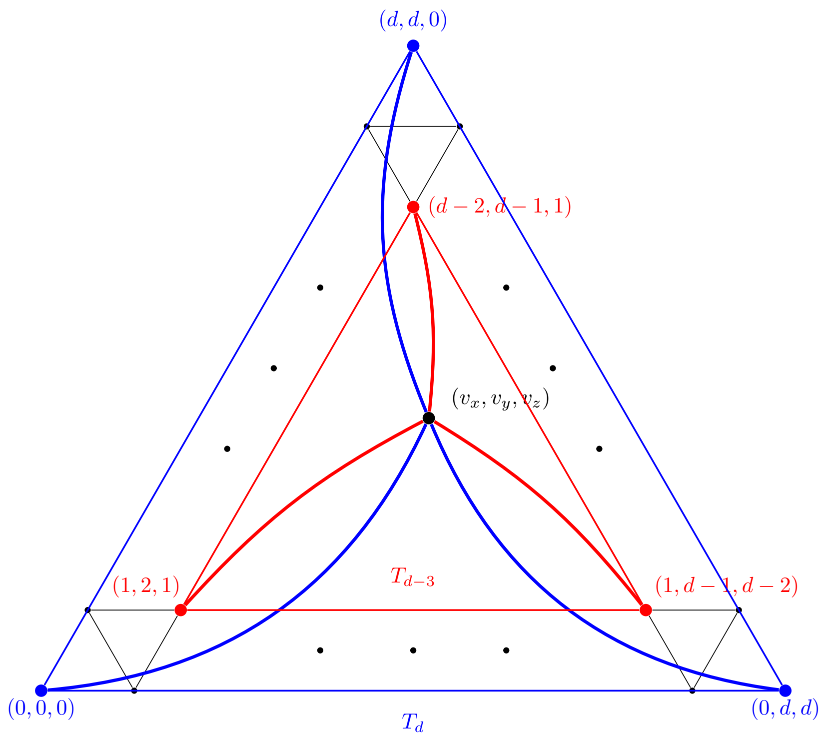


Figure 21. Construction of T_d from T_{d-3} by adding 3 paths parallel to each of its sides.

5.6. Proof of Corollary 1

Recall that we want to show that

$$\gamma_k(T_d) = 1 \quad \text{for all } k \geq r_d = \left\lceil \frac{2d}{3} \right\rceil .$$

Let u be a central vertex of T_d, that is, $ecc(u) = r_d$. Then, for every vertex w of T_d, we have $dist(u, w) \leq r_d$. Hence, the singleton set $\{u\}$ is a distance-k dominating set for every $k \geq r_d$. Consequently, $\gamma_k(T_d) = 1 \quad$ for all $k \geq r_d$.

This result is equivalent to [Harris et al. 2020, Lemma 3.2], where it is expressed as $d \leq \lfloor \frac{3k}{2} \rfloor$ in the context of $(t, 1)$-broadcast domination (with $t = k + 1$). Their proof is based on a geometric argument describing the largest triangular subgraph that fits inside a broadcast hexagon of radius $t - 1$. In contrast, our formulation follows immediately from the radius of T_d.