Filling more gaps on the edge-coloring problem of split graphs

Fernanda Couto[1], Diego A. Ferraz[2], Celina M. H. de Figueiredo[2], Sulamita Klein[2]

[1]Departamento de Computação
Universidade Federal Rural do Rio de Janeiro (UFRRJ) – Nova Iguaçu, RJ – Brazil

[2]Programa de Engenharia de Sistemas e Computação
Universidade Federal do Rio de Janeiro (UFRJ) – Rio de Janeiro, RJ – Brazil

fernandavdc@ufrrj.br,{ferrazda,celina,sula}@cos.ufrj.br

Abstract. *The edge-coloring problem is proven to be* NP*-complete in the general case. However, for several graph classes, this problem remains open. One of these classes is the class of split graphs. Recently, using a partition of this class provided by the* t*-admissibility problem, we classified* $(\sigma = 2)$*-split graphs and a subclass of* $(\sigma = 3)$*-split graphs. In this work, we make a strategic change to a previous algorithm. This change allows us to solve the edge-coloring problem for a larger subclass of* $(\sigma = 3)$*-split graphs.*

Resumo. *O problema da coloração de arestas é provado ser* NP*-completo no caso geral. Entretanto, para diversas classes de grafos, este problema permanece em aberto. Uma destas classes é a classe dos grafos split. Recentemente, utilizando uma partição desta classe fornecida pelo problema da* t*-admissibilidade, classificamos os* $(\sigma = 2)$*-grafos split e uma subclasse dos* $(\sigma = 3)$*-grafos split. Neste trabalho, fazemos uma mudança estratégica em um algoritmo anterior. Tal mudança nos possibilita resolver o problema da coloração de arestas para uma subclasse maior dos* $(\sigma = 3)$*-grafos split.*

1. Introduction

A proper k-edge-coloring of a graph $G = (V, E)$ is an assignment of k colors to the edge set E such that edges incident on the same vertex receive distinct colors. The smallest value of k for which a graph G admits a proper k-edge-coloring is called the chromatic index of G and it is denoted by $\chi'(G)$. A natural lower bound for the chromatic index of any graph G is its maximum degree $\Delta(G)$. Furthermore, Vizing [Vizing 1964] proved that $\Delta(G) + 1$ is an upper bound for the chromatic index of any graph G. Therefore, we are able to classify all graphs in two classes: graphs that admit a proper Δ-edge-coloring, denoted as Class 1 graphs, and graphs which only admit a proper edge-coloring with $\Delta+1$ colors, denoted as Class 2 graphs.

Although we know that every graph is either Class 1 or Class 2, the EDGE-COLORING PROBLEM is a very hard problem and it is proved to be NP-complete in general [Holyer 1981]. There are some graph classes for which this problem is already solved such as bipartite graphs which are Class 1 [Konig 1916], all complete graphs K_n are Class 1 if n is even and Class 2 if n is odd [Behzad et al. 1967] and graphs with a universal vertex ([Behzad et al. 1967], [Plantholt 1981]). Furthermore, there are graph classes for which this problem is open in the general case, such as the split graphs. A graph $G = ((Q, S), E)$ is said to be a split graph if $V(G)$ can be partitioned into a clique Q and an independent set S. Some partial results are known for split graphs in the context of edge-coloring: split graphs with odd maximum degree [Chen et al. 1995] and some intersection graph classes such as split-comparability and split-interval [da Soledade Gonzaga et al. 2023] and split-indifference [de Figueiredo et al. 1999]. And precisely because of the difficulty in tracking down what is missing to solve the edge-coloring problem for split graphs, in a previous paper we proposed a new approach. We consider a partition of split graphs into 3 subclasses given by the T-ADMISSIBILITY PROBLEM[Couto et al. 2025], in which given a graph G and a non negative integer t, we want to know if the graph G admits a spanning tree T in which the greatest distance between two adjacent vertices in G is at most t. The smallest value of t is called the stretch index and is denoted by $\sigma(G)$. Furthermore, if $\sigma(G) = t$ we call G a *$(\sigma = t)$-graph*. It is known that the T-ADMISSIBILITY PROBLEM is polynomial for $t \leq 2$, NP-complete for $t \geq 4$ and is still open for $t = 3$ [Cai and Corneil 1995]. Nevertheless, split graphs are known to be 3-admissible graphs [Panda and Das 2010]. Therefore they can be partitioned into three graph classes: $(\sigma = 1)$, $(\sigma = 2)$ and $(\sigma = 3)$-split graphs. We know that every $(\sigma = 1)$-split graph is Class 1 since they are trees. In a previous work, we classified $(\sigma = 2)$-split graphs [Couto et al. 2025]. So, in order to fully classify split graphs, it remains to classify $(\sigma = 3)$-split graphs, and recently we classified a subclass of this class [Couto et al. 2026]. In this work, we expand the scope of the algorithm developed in [Couto et al. 2026] through a strategic change in the construction of the saturated graph, a crucial structure used by Plantholt in his seminal technique for coloring the edges of graphs with a universal vertex [Plantholt 1981].

2. Preliminaries

The main goal of this work is to upgrade a previous algorithm we showed in [Couto et al. 2026].

Theorem 2.1 ([Couto et al. 2026]) *Let G be a $(\sigma = 3)$-split graph. If there is a vertex $s_1 \in S(G)$ adjacent to a Δ-vertex and $d(s_1) = \dfrac{\Delta(G)}{2}$, then G is Class 1.*

The proof for this sufficient condition is given by an algorithm which colors $(\sigma = 3)$-split graphs that satisfy the hypothesis. The algorithm acts as an extension of Plantholt's seminal result which establishes a necessary and sufficient condition for graphs with a universal vertex and odd number of vertices to be Class 2.

Theorem 2.2 *[Plantholt 1981]] Let G be a graph with an universal vertex and an odd number of vertices. Then G is Class 2 if, and only if, G is subgraph-overfull.*

In general therms, given a $(\sigma = 3)$-split graph G, we prove there is an induced subgraph H with the same maximum degree which is a $(\sigma = 2)$-split graph. With this subgraph in hand, we construct a saturated graph H^*. First, note that the saturated graph, defined by Plantholt, is a graph with $\Delta(H^*) \cdot \left\lfloor \dfrac{|V(H^*)|}{2} \right\rfloor$ edges, and no special condition to be satisfied, but since we deal with Split graphs with a vertex with degree $\dfrac{\Delta(G)}{2}$, we presented in [Couto et al. 2026] an algorithm that constructs a saturated graph, denoted by

H^*, with a particular structure: it remains a split graph with a clique Q of size $|V(H)|-1$ and an independent set of size 1.

After, the graph H^* and, consequently, the graph H are edge-colored, we bring back the elements of $G \setminus H$. In order to color these remaining edges, we extend the edge-coloring provided by Plantholt to the graph H^* to the edges of G that are not colored yet. Basically, if there is an edge qw in $E(G \setminus H)$ (not colored yet) and a colored edge qs that belongs to H^* but does not belong to G, then we apply the color assigned to qs by Plantholt's method to the edge qw, where $q \in Q(H)$, $s \in S(H)$ and $w \in V(G \setminus H)$. Note that this assignment is not necessarily proper, and, in order to provide a proper one, we developed a tool called *Color Trail*.

The Color Trail contains all edges incident to a vertex $w \in V(G \setminus H)$ with color conflicts. Each edge $qw \in E(G \setminus H)$ is represented as a solid edge. If there is an edge $qq' \in E(H^* \setminus H)$ and the color assigned to qq' by Plantholt's algorithm is assigned to qw by our extension, then we say that qw is paired with qq', which is represented in the Color Trail by a dashed line. If the color of the edge qw does not come from an edge of $E(H^* \setminus H)$, then in the Color Trail there is no dashed edge paired with qw. There is only one rule to order the edges in the construction of the Color Trail: for each color, the rightmost edge is the one with no dashed pair, if one exists [Couto et al. 2026].

If two edges wx and wy receive the same color by extension, then there are two different vertices of H^*, say a, b with $a, b \neq s_1$, such that ax and by receive the same color assignment in H^*. Moreover, the edges $ay, bx \in E(G)$, otherwise, wx and wy would have been colored using the colors that were incident either to a or either to b, and the edge-coloring would be proper. Thus, it is possible to swap the colors of the edges ay and wy, since the color used at wy is a missing color of vertex a. Note that the color conflict at w is solved once both colors were assigned to vertex a by Plantholt's edge-coloring. We prove that these color swaps are possible with the following lemma.

Lemma 2.1 ([Couto et al. 2026]) *If there is a color conflict between edges wx_i and wx_j there is a vertex $x'_{j^*} \neq s_1$ in the independent set of G which has such a color as a missing color, for $1 \leq j^* < i$ in the color trail.*

As stated by Lemma 2.1, the swaps can be done only when a vertex is adjacent to all vertices on its right in the Color Trail. However, the vertex s_1 is not a universal vertex and thus, when $d(s_1) < \frac{\Delta(G)}{2}$, the vertex s_1 will receive new edges on the graph H^* that are not on G and this can lead us to a situation where no swaps are possible. In order to improve our previous result, we allow the vertex s_1 to have degree less than or equal to $\frac{\Delta(G)}{2}$ whenever it is possible to verify the conditions that will be described in the next section, as they guarantee that vertex s_1 will not participate in the Color Trail.

3. Results

In this section, we present a new version of the algorithm which builds the saturated graph H^*. This new version has the potential to increase the range of input graphs in which we perform a proper edge-coloring with Δ colors. If the number of edges we need to add to s_1 in order to construct the saturated graph is less than or equal to the number of vertices in the graph which are not adjacent to s_1 in the graph $G \setminus H$ and do not have any neighbors in $G \setminus H$, we can make s_1 adjacent to these vertices and thus, we can edge-color more graphs such that $\delta < \dfrac{\Delta(G)}{2}$. If this is not possible, then we complete the necessary edges for s_1 by making this vertex adjacent to vertices which will not put s_1 on the Color Trail because they are adjacent to other vertices in H^* which make the extension possible from these vertices. And finally, if none of these conditions is satisfied, we can not guarantee a proper edge-coloring using the Color Trail, so we stop our method. The following lemma describes this idea.

Lemma 3.1 *Let $Q' = \{q_i | q_i \in \overline{N_Q(s_1)}$ such that $N_{G\setminus H}(q_i) = \emptyset\}$. If $|Q'| \geq \frac{\Delta(G)}{2} - d_G(s_1)$, then s_1 can be made adjacent to vertices of Q', avoiding vertices with $N_{G\setminus H}(q_i) \neq \emptyset\}$. Let $Q'' = \{q_i | q_i \in \overline{N_Q(s_1)}$ such that $d_{G\setminus H}(q_i) \leq d_{H[E']}(q_i)$. If $\frac{\Delta(G)}{2} - (d_G(s_1) + |Q'|) \leq |Q''|$, then s_1 can be made adjacent to vertices of Q'', which eliminates s_1 from the Color Trail.*

The Algorithm 1 shows the new version of the Algorithm described in Section 2.

Algorithm 1: Refined construction of the special saturated graph H^*

Data: A graph $H = ((Q, S), E)$ with a vertex s_1, such that $d(s_1) \leq \frac{\Delta(H)}{2}$ and a universal vertex. Furthermore, $\Delta(H)$ is even.

if $|Q| < |V(H)| - 1$ **then**
  Add to Q each vertex of S but s_1, obtaining a clique Q^* of size $|V(H)| - 1$
if $|E^*| < \Delta(H) \cdot \left\lfloor \frac{|V(H)|}{2} \right\rfloor$, $E^* = E \cup E'$ *where E' are the edges added to obtain Q^** **then**
  Let $Q' = \{q_i | q_i \in \overline{N_Q(s_1)}$ such that $N_{G\setminus H}(q_i) = \emptyset\}$
  if $|Q'| \geq \frac{\Delta(G)}{2} - d_G(s_1)$ **then**
    Make s_1 adjacent to $\left(\frac{\Delta(G)}{2} - d_G(s_1)\right)$ vertices of Q'
  else
    Make s_1 adjacent to vertices of Q'
    Let $Q'' = \{q_i | q_i \in \overline{N_Q(s_1)}$ such that $d_{G\setminus H}(q_i) \leq d_{H[E']}(q_i)\}$
    if $\frac{\Delta(G)}{2} - (d_G(s_1) + |Q'|) \leq |Q''|$ **then**
      Make s_1 adjacent to $\left(\frac{\Delta(G)}{2} - (d_G(s_1) + |Q'|)\right)$ vertices of Q''
    else
      return *false*
return $H^* = ((Q^*, S^* = \{s_1\}), E^*)$

If the first condition described in Lemma 3.1 is satisfied, the connections between the vertex s_1 and its neighbors in the saturated graph can be made in a convenient way such that no color swap is necessary from the vertex s_1 in the Color Trail. If the second condition is satisfied, then there are not vertices in Q' in order to complete the adjacencies of the vertex s_1. Thus, we complete these connections using the vertices of Q''. These vertices will not bring the vertex s_1 to the Color Trail, so the color extensions can be made using these other vertices. The results presented in the previous section consider a subclass of the $(\sigma = 3)$-split graphs where the $(\sigma = 2)$-subgraph H has a vertex s_1 such that $d(s_1) = \dfrac{\Delta(G)}{2}$. We established this last condition because the Color Trail performs color swaps using the new edges which were added in the saturated graph. And since the vertex s_1 is not adjacent to $\dfrac{\Delta(G)}{2}$ vertices of $Q(H)$, if the edge connecting s_1 to any of these vertices of $Q(H)$ is necessary of a color swap, the Color Trail could not resolve the problem. However, we observed that if we guarantee that any of the conditions described in Lemma 3.1 is satisfied, then it is possible to perform a proper Δ-edge-coloring of a larger subclass of $(\sigma = 3)$-split graphs.

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A. Appendix

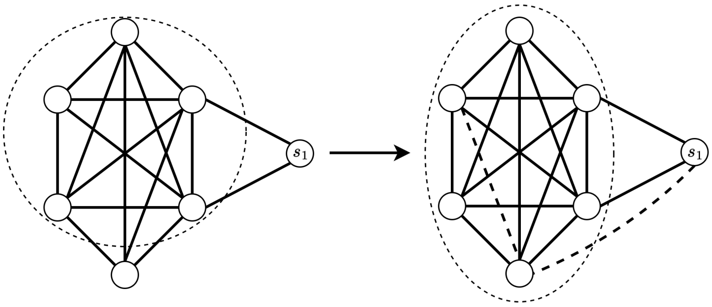

Figura 1. Construction of a saturated graph H^*. The dashed ellipses denote the cliques of each graph and the thicker dashed lines denote added edges.

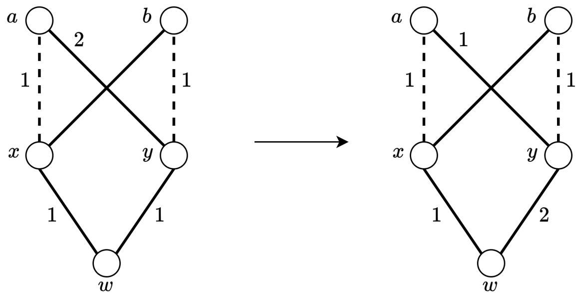

Figura 2. On the left there are two conflicting edges: wx, wy colored with color 1. On the right, wx remains colored with color 1 and wy and ya swapped colors.