Hardness results on parity subdivisions in digraphs*

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Abstract. *The paper presents computational complexity results on the problem of parity subdivision in digraphs. We say that a vertex is* small *if its total degree (the sum of its in-degree and its out-degree) is at most* 3 *and it does not have in-degree or out-degree equal to* 3*; otherwise it is called* big*. We prove that the Parity* (s,t)*-Path problem is NP-complete when restricted to digraphs with only small vertices. With this result, we demonstrate that Parity Subdivision is also NP-complete. We conjecture that Parity* F*-Subdivision is NP-complete for any digraph* F *with at least two connected big vertices.Finally, we discuss cases in which the problem is solvable in polynomial time, such as parity path subdivision, 3-odd-cycle, and spiders.*

1. Introduction

Throughout this paper, we use standard graph and digraph terminology. For a digraph D, we denote by $V(D)$ its set of vertices and by $A(D)$ its set of arcs. The *degree* of a vertex v in a digraph is the sum of its in-degree and out-degree.

The set of results in Graph Minor Theory are well known for its importance in the field. One of the problems explored in this series is the one of finding subdivisions: A *subdivision* of a graph $H = (V, E)$ is a graph obtained from H by replacing each edge $(a, b) \in E(H)$ with a path (which may be the edge itself) between a and b internally disjoint from the rest of the graph. In [Robertson and Seymour 1995], the authors showed that finding a subdivision of a fixed graph H in a graph G can be done in polynomial time.

Later, Kawarabayashi, Reed and Wollan [Kawarabayashi et al. 2011] approached the problem with parity conditions. They presented a polynomial-time algorithm to find a subdivision of H in G such that each path respects a parity received in the input.

Many other variations of the subdivision problem were studied. When looking at directed graphs, the news are not so good, concerning complexity: for a general digraph F, finding a subdivision of F in a digraph D is NP-complete [Bang-Jensen et al. 2015]. In subdivisions of digraphs, the arcs are replaced by directed paths, preserving the same direction as the original arc. Many cases for which this problem is NP-complete, as well as special types of digraphs for which it is polynomial-time solvable, can be found in [Bang-Jensen et al. 2015, Havet et al. 2018].

In this work, following the approach of Kawarabayashi, Reed and Wollan [Kawarabayashi et al. 2011], we study the subdivision problem with parity conditions in digraphs. More specifically, we approach the following problem:

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PARITY F-SUBDIVISION (F-PS)
Input: A digraph D and a parity function $J : A(F) \to \{0, 1\}$.
Question: Does D have an F-subdivision that respects J?

Informally, an F-subdivision of D *respects* J when each path in D that corresponds to an arc uv of F has length whose parity equals $J(uv)$ (a formal definition is given in Section 2).

In this paper, we establish that Parity (s,t)-Path problem is NP-complete even when restricted to digraphs with only small vertices. A vertex is defined as *small* if its degree is at most 3, and it does not have in-degree or out-degree equals 3. Building on this, we prove that Parity $F_{3\to4}$-Subdivision is NP-complete. Afterwards, we discuss some cases where F-PS is polynomial-time solvable.

2. Hardness results of problems with parity

In this section, we present hardness results of some problems related to the problem of Parity subdivision.

2.1. Parity paths on Digraphs

Finding a path between two given vertices in a graph is widely known to be in polynomial time for both undirected and directed graph as it can be solved by a breadth-first search or depth-first search. When we introduce the parity of the path as a parameter (i.e. the length of the path is specified to be even or odd), we have different complexities for undirected and directed graphs. This problem is polynomial-time solvable for undirected graph but NP-complete for directed graphs [LaPaugh and Papadimitriou 1984].

PARITY (s, t)-PATH
Input: A digraph D, two vertices $s, t \in D$, and $j \in \{0, 1\}$.
Question: Is there a path from s to t whose length has the same parity of j?

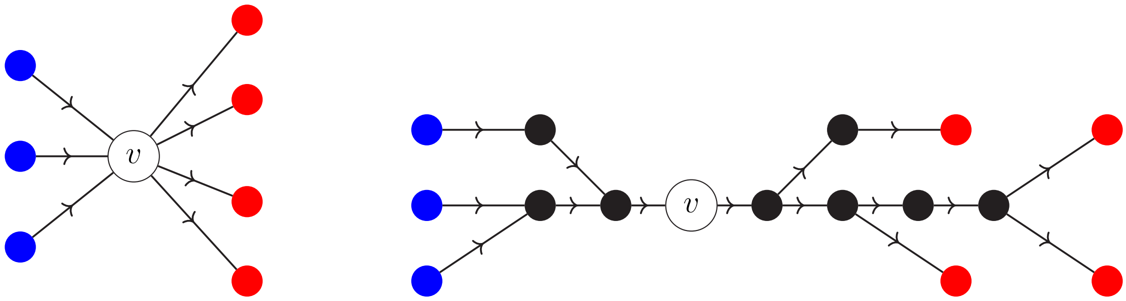
We know that PARITY (s, t)-PATH is NP-complete when $j = 0$ [LaPaugh and Papadimitriou 1984, Theorem 2], and this result can be easily extended to show the following:

Theorem 1. [LaPaugh and Papadimitriou 1984] The PARITY (s, t)-PATH problem is NP-complete.

An *out-arborescence* is a rooted tree in which all vertices have in-degree 1, except the root. We say it is a *switching out-arborescence* if the root has out-degree 1, the leaves have out-degree 0, and all other vertices have out-degree 2. Analogously, we define *in-arborescence* and *switching in-arborescence*.

Following a similar idea of [Bang-Jensen et al. 2015], we consider below a restriction on the PARITY (s, t)-PATH problem where the input is a digraph with only small vertices. Next we show that this variation of the problem is also NP-complete.

PARITY (s, t)-PATH-SMALL
Input: A digraph D with only small vertices, two vertices $s, t \in D$, and $j \in \{0, 1\}$.
Question: Is there a path from s to t whose length has the same parity of j?



(a) Digraph D **(b)** Digraph D'

Figure 1. Digraph D' after swapping $N^+(v)$ and $N^-(v)$ by an in- and out-arborescence with root v and leaves $N^+(v)$ and $N^-(v)$, respectively.

Theorem 2. The PARITY (s,t)-PATH-SMALL problem is NP-complete.

Proof. We define a reduction from PARITY (s,t)-PATH. Given a digraph D we obtain a digraph D' as follows: for each vertex v, replace all the arcs leaving v by a switching out-arborescence with root v, leaves corresponding to the out-neighbors of v in D, and making sure that a path from the root to a leave has odd length. Also replace all the arcs entering v by a switching in-arborescence with root v, leaves corresponding to the in-neighbors of v in D, and making sure that a path from a leaf to the root has odd length. Note that every vertex in D' must be small by our construction. Our construction also guarantees that each arc xy in D is replaced by an odd path from x to y in D'. So there is an odd path between s and t in D if and only in there is an odd path between s and t in D'. $\square$

2.2. Subdivision with Parity

An F-subdivision respects J when the paths in D that correspond to arcs in F agree with the parity of the arc, i.e. the length of the path that corresponds to arc uv must have parity $J(uv)$. Now consider the digraph $F_{3\to4}$ shown in Figure 2, and notice that vertices x and y are big.

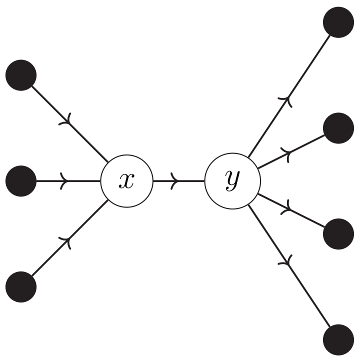


Figure 2. Digraph $F_{3\to4}$

Theorem 3. PARITY $F_{3\to4}$-SUBDIVISION is NP-complete.

Proof. We define a reduction from the PARITY (s,t)-PATH-SMALL problem, which is NP-complete by Theorem 2. Given a digraph D with only small vertices, given vertices $s, t \in V(D)$, and a parity j, we define a digraph $D' = (V', A')$ as follows: we add 7 new vertices to V', so $V' = V \cup \{i_1, i_2, i_3, o_1, o_2, o_3, o_4\}$, and A' includes all arcs in A plus arcs from $\{i_1, i_2, i_3\}$ to s and from t to $\{o_1, o_2, o_3, o_4\}$ as depicted in Figure 3. We also associate the following parities to arcs in $F_{3\to4}$: $J(xy) = j$ and $J(e) = 1$ for all $e \in A(F_{3\to4}) - xy$.

($\Rightarrow$) Assume there is a path from s to t with parity j in D. This path will correspond to arc xy of the $F_{3\to4}$-subdivision in digraph D'.

($\Leftarrow$) Assume that D' has an $F_{3\to4}$-subdivision. Observe that s, t are the only big vertices in D' by our construction, so they must correspond to vertices x and y, respectively. Moreover, the path that corresponds to arc xy must be fully inside the dotted area, and therefore there is a path from s to t of parity j in D.

$\square$

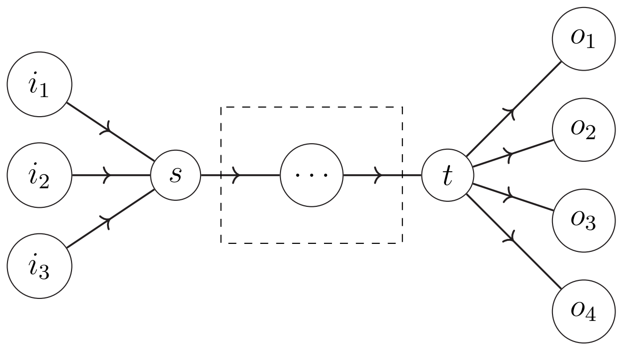


Figure 3. Digraph D' obtained from D.

Conjecture 1. PARITY F-SUBDIVISION is NP-complete for any digraph F with at least two big vertices that are connected.

3. A word on polynomial cases

Parity subdivision is in P for the following graphs F: directed paths, 3-Odd-cycle and spiders [Maia and Melo 2022]. A 3-Odd cycle is a direct cycle with length 3 where, for all arcs e, $J(e) = 1$. Each of these cases has a simple underlying idea. Finding a Parity Path Subdivision reduces to finding a directed path of minimum length such that the path qualifies as a valid subdivision.

Finding a 3-Odd-Cycle Subdivision reduces to finding an odd cycle in the graph, which takes polynomial time. A spider is an oriented graph with a root vertex r and disjoint paths starting in r or ending in r. Finding a Parity Spider Subdivision reduces to finding a spider whose paths meet the minimum length required to form a valid subdivision. The cases for odd cycles bigger than 3 and even cycles remains open.

We recently discovered an interesting case, described below:

Theorem 4. F-PS is polynomial-time solvable in digraphs with maximum circumference 2.

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