Lower Bounds on the Convergence to Nash Equilibrium in the Bin Packing Game

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Abstract. *This paper analyzes the convergence time to Nash equilibrium in the selfish bin packing game under the elementary stepwise system (ESS). By using structural dualities between bin packing and load balancing games, we establish the first non-trivial lower bounds for this problem. We demonstrate that convergence can require exponential time in the general case, while proving a tight bound of $\Theta(n^2)$ steps for instances with uncapacitated bins under the best-response strategy. Our results introduce proof techniques that depart from traditional potential functions.*

1. Introduction

In this work, we study a *bin packing game* where each item is controlled by a selfish player and a central authority is absent. We analyze the number of steps (*i.e.*, updates of actions) for the system to reach a *Nash equilibrium*. In our model, we assume the *elementary stepwise system* (ESS), *i.e.*, at each step only one item updates its action. In this bin packing game, we have n items and m bins. All bins have the same capacity C and a cost equal to one; each item i has an integer size w_i. Let ℓ_j be the load of bin j, *i.e.*, the total size of items assigned to bin j. An item i assigned to a bin j pays the fraction of the load it is using, or w_i/ℓ_j. Since i is selfish and wants to minimize its cost, it migrates to another bin j' if $w_i + \ell_{j'} \leq C$ and $w_i/(w_i + \ell_{j'}) < w_i/\ell_j$ (*i.e.*, $w_i + \ell_{j'} > \ell_j$). That is, i moves from j to j' if it fits in j' and its new cost is strictly smaller. A Nash equilibrium is a feasible packing where no player can reduce its cost by moving to another bin.

Related Work: Bilò [Bilò 2006] pioneered the study of the bin packing game in the ESS model, establishing an $O(P^2)$ upper bound for convergence to a Nash equilibrium, where P denotes the sum of item sizes, and defining bounds for the Price of Anarchy. The tightness of this $O(P^2)$ convergence bound remained an open question, which serves as the primary motivation for our work, the derivation of a non-trivial lower bound. The game-theoretic study of bin packing remains a niche field limited to four foundational papers. Following Bilò, Epstein and Kleiman [Epstein and Kleiman 2008] refined the Price of Anarchy bounds, while Yu and Zhang [Yu and Zhang 2008] demonstrated that computing a pure Nash equilibrium is possible in polynomial time via a centralized algorithm. Subsequently, Miyazawa and Vignatti [Miyazawa and Vignatti 2009, Miyazawa and Vignatti 2011] explored convergence time within distributed settings, providing logarithmic and polynomial bounds. This problem is closely related to load balancing; Bilò [Bilò 2006] leveraged potential functions and parallels identified by Even-dar et al. [Even-Dar et al. 2007] to analyze convergence. However, while Even-dar et al. established both upper and lower bounds for load balancing convergence, a similar comprehensive analysis is currently absent for the bin packing game. In an alternative configuration, specifically the variant involving equal cost-sharing of the bins, a study on convergence time was conducted by Ma et al. [Ma et al. 2013], with results later improved by Dósa and Epstein [Dósa and Epstein 2018]. These results are not directly comparable to ours, as our model employs a proportional cost-sharing mechanism.

Our Results: We first use the similarities between the bin packing and the load balancing games to show an exponential lower bound on the convergence time to Nash equilibrium in selfish bin packing. We also show that, using the best-response strategy and bins without capacity limitations, in $\Theta(n^2)$ steps the bin packing game reaches the Nash equilibrium. It is worth noting that our results are the first non-trivial bounds for the problem, and our proof techniques are different from those in [Bilò 2006, Even-Dar et al. 2007] and from other papers regarding convergence time to Nash equilibrium.

2. Model Description

Next, we describe the model for the bin packing game. We also outline the load balancing game, as several results from this problem are utilized in Section 3.

The *bin packing game* consists of m bins, each with capacity C and cost 1, and n items with *sizes* $w_1, \ldots, w_n$. Let $[n]$ be the set of items and $[m]$ the set of bins. A function $A : [n] \to [m]$ represents a configuration (or state) of the game if, for every bin j, its *load* ℓ_j does not exceed C, where $\ell_j = \sum_{i \in [n]: A(i)=j} w_i$. Each item is controlled by a selfish player seeking to minimize their cost. Throughout this work, we use the index i interchangeably to denote both the item and its corresponding player. When player i is assigned to bin j, the cost incurred is w_i/ℓ_j. Consequently, player i will migrate from bin j to another bin $j' \neq j$ if $w_i + \ell_{j'} \leq C$ and $w_i/(w_i + \ell_{j'}) < w_i/\ell_j$ (*i.e.*, $w_i + \ell_{j'} > \ell_j$).

The *load balancing* model is similar to the bin packing game described above, with two primary differences: (1) $C = \infty$, and (2) when a player i uses bin j, the cost incurred is simply the total load ℓ_j. In this case, player i will migrate to a bin $j' \neq j$ if $\ell_{j'} + w_i < \ell_j$. We let $k \leq n$ denote the number of distinct item sizes and S denote the sum of these distinct sizes.

3. An Exponential Lower Bound

The open question posed by Bilò [Bilò 2006] motivates our search for a non-trivial lower bound in the selfish bin packing game. In this section, we use the structural similarities between bin packing and load balancing games to establish an exponential lower bound on the time to reach a Nash equilibrium.

In both games, an assignment is defined as a mapping of n items to m bins. The fundamental distinction lies in the players' incentives for migration: an item i migrates from bin j to bin j' in the bin packing game if $\ell_{j'} + w_i > \ell_j$, whereas in the load balancing game, migration occurs if $\ell_{j'} + w_i < \ell_j$. Based on this observation, we directly arrive at the following proposition.

Proposition 1. *Let A be an assignment of items to bins, and let A' be an assignment obtained from A when a player moves their item to a different bin according to the migration incentives of the load balancing game. It follows that the reverse migration – moving* *from A' back to A – is a valid migration according to the migration incentives of the bin packing game.*

By repeatedly applying the logic of Proposition 1, we obtain the following corollary.

Corollary 2. *Let $(A_1, A_2, A_3, \ldots, A_N)$ be a sequence of assignments resulting from valid migrations in a load balancing game, where each A_{i+1} is obtained from A_i by moving an item to a different bin according to player incentives. It follows that the reverse sequence $(A_N, A_{N-1}, A_{N-2}, \ldots, A_1)$ constitutes a sequence of valid migrations in the bin packing game, provided that no migration exceeds the bin capacities.*

Corollary 2 suggests that any lower bound on the number of steps to a Nash equilibrium in the load balancing game also serves as a lower bound on the number of steps to the Nash equilibrium for the bin packing game, provided that no migration exceeds the bin capacities.

Next, Theorem 3 presents a lower bound established by Even-Dar et al. [Even-Dar et al. 2007] (see Theorem 5.5 in the cited work). This theorem assumes a minimum-weight best-response strategy; that is, among all unsatisfied players, those with the lightest items are prioritized to perform their best-response migration.

Theorem 3 ([Even-Dar et al. 2007])**.** *There is an instance of the load balancing game with n items and m bins where the* minimum weight best-response *strategy reaches the Nash equilibrium in at least $\left(\frac{n}{(m-1)^2}\right)^{(m-1)}$ steps.*

Theorem 3 (specifically Theorem 5.5 in [Even-Dar et al. 2007]) establishes that under the minimum-weight best-response strategy, the number of steps required to reach a Nash equilibrium is exponential for certain game instances.

Combining Theorem 3 with Corollary 2 implies the existence of instances and migration sequences in the bin packing game that similarly require exponential time to reach a Nash equilibrium. To construct such instances, we follow the methodology in the proof of Theorem 3 (see [Even-Dar et al. 2007]), ensuring that the item weights do not exceed the bin capacities. This leads to the following result:

Theorem 4. *There exists an instance of the bin packing game with n items and m bins, and a sequence of migrations therein, such that reaching a Nash equilibrium requires at least $\left(\frac{n}{(m-1)^2}\right)^{(m-1)}$ steps.*

4. A Tight Bound for Uncapacitated Bins using Best-Response

The result of Theorem 4 partially addresses the open question posed by Bilò [Bilò 2006] regarding a non-trivial lower bound for the bin packing game. Note that while Theorem 3 is formulated specifically for best-response migrations, Theorem 4 does not share this constraint. This distinction arises because the "reverse migration" of a best-response in the load balancing game – though a valid move in the bin packing game – is not necessarily a best-response within that context. Consequently, a separate analysis of best-response migrations in the bin packing game is required.

In Theorem 5, we establish a quadratic lower bound for bin packing game instances featuring two bins, unlimited capacities, and the best-response strategy. The proof follows a similar logic to Theorem 5.6 in [Even-Dar et al. 2007].

Theorem 5. *In the bin packing game with n items and 2 bins of* unlimited *capacity, there exists a set of item weights and a sequence of* best-response *migrations that requires $\Omega(n^2)$ steps to reach a Nash equilibrium.*

Proof. We partition the items into $t = n/2$ classes, where each class contains two items of weight 3^i for $i \in \{1, \ldots, t\}$. Note that the weight of an item in class i is strictly greater than the sum of the weights of all items in classes $1, \ldots, i-1$. Initially, the items are distributed such that the two items of each class are in different bins. The migration follows this rule: at each step, select the lightest items that have an incentive to migrate (at most two such items exist), and move an item from the more heavily loaded bin (breaking ties arbitrarily). Under this rule, it can be observed that $\Omega(n^2)$ steps are required before the system reaches a Nash equilibrium. $\square$

For a given assignment of items to bins, we define the heaviest bin as the bin with the largest total load. Lemma 6 provides a bound on the number of times the identity of the heaviest bin changes – that is, the number of instances in which a different bin assumes the maximum load.

Lemma 6. *In the bin packing game where the bins are* uncapacitated *and the players use the best-response strategy, the heaviest bin changes at most k times.*

Proof. Let j be one of the heaviest bins at a given time. In order for j to no longer be the heaviest bin, an item assigned to j must migrate. Let i be the item that migrates from j. As the strategies are chosen according to the best-response, i chooses the second heaviest bin to migrate to, say, bin j'. Let ℓ_j and $\ell_{j'}$ be, respectively, the weights of bins j and j' before i migrates. Note that the other bins have weights of at most $\ell_{j'}$. After the migration, bin j' increases its weight to $\ell_{j'} + w_i$ and becomes the heaviest bin. Thus, the difference in weight between j' and the other bins, except for bin j, becomes at least w_i. The weight difference between j' and j becomes $\ell_{j'} + w_i - (\ell_j - w_i) \geq w_i$. Hence, j', which is now the heaviest bin, has a weight difference of at least w_i with respect to all bins. This weight difference grows with each migration, because the items will migrate to bin j'. Therefore, if a player moves its item from j' (and consequently the heaviest bin changes again), this item should be strictly greater than w_i since, as noted earlier, the weight difference of the bins is at least w_i. Thus, when the heaviest bin changes, it is because of a migration of an item that is heavier than the item that provoked the previous change of the heaviest bin. As we have at most k different weights, the result follows. $\square$

Theorem 7. *In the bin packing game where the bins are* uncapacitated *and the players use the best-response strategy, the Nash equilibrium is reached in at most $kn \leq n^2$ steps.*

Proof. Due to the best-response strategy, items not assigned to the heaviest bin migrate to it. Thus, if the heaviest bin does not change, there are at most n migrations before the Nash equilibrium is reached. From Lemma 6, there are at most k changes to the heaviest bin; therefore, the result follows. $\square$

Due to Theorem 5, the result of Theorem 7 cannot be asymptotically improved.

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