NP-Completeness of Independent Locating-Dominating Sets in Planar Bipartite and Subcubic Graphs

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Abstract. *The* Independent Locating-Dominating Set *(ILD) problem consists of determining whether a graph admits a vertex subset that is simultaneously independent, dominating, and able to uniquely identify all non-selected vertices. In this paper, we study the computational complexity of the ILD problem in restricted graph classes and prove that it remains NP-complete for planar bipartite graphs and planar triangle-free subcubic graphs, via polynomial-time reductions from variants of the 3-SAT problem.*

1. Introduction

Consider a scenario in which a graph G models a physical facility or a multiprocessor communication network, where detection devices with limited sensing capabilities are installed at selected vertices. These devices are able to detect the presence of an undesirable event, such as an intruder, malfunction, or abnormal activity, either at their own location or within their open neighbourhood, but they are not able to distinguish between different vertices within that neighbourhood. Since the installation and maintenance of such devices are costly, it is desirable to minimize their number while ensuring that the location of any event can be uniquely identified.

This setting leads to the study of locating-dominating sets, which extend the classical notion of domination by requiring that every vertex not in the selected set is uniquely identified by its neighbourhood within the set [Chakraborty et al. 2025]. The modelling of detection and identification problems in networks using domination-type concepts has been explored in [Rall and Slater 1984]. In many practical situations, additional constraints arise. For instance, interference between devices may prevent placing detectors at adjacent vertices, which imposes that the selected set must be independent. This motivates the concept of independent locating-dominating sets [Slater and Sewell 2018].

Throughout this paper, let $G = (V(G), E(G))$ be a non-trivial, simple, undirected, and connected graph. The *open neighbourhood* of a vertex $v \in V(G)$ is defined as $N(v) = \{u \in V(G) \mid uv \in E(G)\}$. A pair of vertices $u, v \in V(G)$ are called *false twins* if $N(u) = N(v)$. For notational convenience, for any positive integer k, we write $[k] = \{1, 2, \ldots, k\}$.

A vertex subset $S \subseteq V(G)$ is *dominating* if every vertex in $V(G) \setminus S$ is adjacent to at least one vertex of S. The set S is a *locating-dominating set* (or *LD set*) if it is dominating and, for every pair of distinct vertices $u, v \in V(G) \setminus S$, it holds that $N(u) \cap$

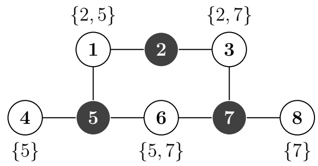


Figure 1. An *ILD*-set $\mathrm{S} = \{2, 5, 7\}$ for a graph G.

$S \neq N(v) \cap S$. The minimum cardinality of such a set is denoted by $LD(G)$. A set $S \subseteq V(G)$ is *independent* if no two vertices of S are adjacent. An *independent locating-dominating set* (or *ILD set*) is a subset that is both independent and locating-dominating. The minimum cardinality of an ILD set in G is denoted by $ILD(G)$. In Figure 1, we depict an illustrative instance of an ILD set S, in which, for every vertex $v \in V(G) \setminus S$, the corresponding set $N(v) \cap S$ is explicitly indicated. The independent locating-dominating set problem (ILD) is defined as follows:

INDEPENDENT LOCATING-DOMINATING SET (ILD).
Instance: A graph G.
Question: Does G admit an independent locating-dominating set?

Not every graph admits an independent locating-dominating set, which naturally motivates the study of structural conditions ensuring the existence of such sets. This non-existence phenomenon even arises in very restricted graph classes, such as cycle graphs. For example, the cycles C_3 and C_4 do not admit ILD sets. The notion of an independent locating-dominating set was introduced by Slater and Sewell [Slater and Sewell 2018], who also proved that the ILD problem is NP-complete in general graphs, thereby providing fundamental complexity-theoretic foundations for its subsequent study.

From an algorithmic standpoint, domination-type problems are well known for their inherent computational intractability. Several variants, including locating-dominating sets and identifying codes, have been extensively analysed due to their relevance to applications such as network monitoring, fault detection, and optimal resource placement. For many of these problems, NP-completeness persists even under strong structural restrictions on the input graph, for instance when the graph is planar of bounded maximum degree, or bipartite [Müller et al. 2009, Foucaud 2015, Rahbani et al. 2019]. In contrast, for the ILD problem, only a limited number of results are currently available concerning its complexity under such constraints [Lemos et al. 2025a, Lemos et al. 2025b].

In this work, we study the computational complexity of ILD on structurally restricted graph classes. We show that the problem remains NP-complete when confined to planar bipartite graphs, as well as to planar triangle-free subcubic graphs. Our proofs are based on polynomial-time reductions from suitably chosen variants of the satisfiability problem, employing carefully constructed gadgets that encode variable assignments and clause satisfaction while preserving planarity and degree bounds. These results demonstrate that the computational hardness of ILD is robust and persists even under strigent structural constraints on the input graph.

2. Complexity results

Given the NP-completeness result of Slater and Sewell, we now turn our attention to restricted graph classes. A key observation is that, in any ILD-set S, all vertices belonging to a set of false twins must be included in S. This property plays an important role in our reductions. In the following theorem, we show that ILD remains NP-complete even when restricted to planar bipartite graphs (PBILD), by means of a reduction from the well-known decision problem PLANAR 3-SAT.

Theorem 2.1. *Deciding, for a given planar bipartite graph G, if G has an* ILD*-set is an NP-complete problem.*

Proof. Clearly ILD is in NP. We now show that PLANAR 3-SAT is reducible in polynomial time to the problem PBILD. Let ϕ be an instance of PLANAR 3-SAT, where $U = \{u_1, u_2, \dots, u_n\}$ is the variables set and $C = \{c_1, c_2, \dots, c_m\}$ is the clauses set. For each $u_i \in U$, add a copy of graph H_i shown in Figure 2(a). For each $c_j \in C$, add a copy of graph Z_j shown in Figure 2(b). Then, each vertex α_j of clause c_j is made adjacent to the three literal vertices in c_j. The resulting graph G has order $|V(G)| = 12n + 4m$ and size $|E(G)| = 14n + 4m$, and can be constructed in polynomial time.

Clearly, G is planar since ϕ is an instance of PLANAR 3-SAT. Furthermore, G is bipartite because each gadget H_i and c_j is bipartite, and u_i and $\overline{u_i}$ are in the same part of $V(G)$, say A, and α_j is in the other part, say B. We now prove the equivalence between (i) the existence of an *ILD*-set in G and (ii) the satisfiability of ϕ. Observe that for every *ILD*-set S, say of H_i, $|S \cap \{u_i, \overline{u_i}\}| = 1$. To prove that (i) is sufficient for (ii), assume that ϕ has a satisfying truth assignment. Let $D_1 = \{u_i, g_i \mid u_i \text{ is True}\}$ and $D_2 = \{\overline{u_i}, e_i \mid u_i \text{ is False}\}$. Let $S = \{a_i, f_i, p_i, k_i\} \cup D_1 \cup D_2 \cup \{x_j, y_j \mid 1 \leqslant j \leqslant m\}$. It is straightforward to verify that S is an *ILD* set of G.

To prove that (ii) is sufficient for (i), assume that G has an *ILD*-set S. Observe that $\{a_i, f_i, p_i, k_i\} \subseteq S$ for all $i \in [n]$ and $\{x_j, y_j\} \subseteq S$ for all $j \in [m]$, since all these vertices are false twins. Moreover, S cannot contain both u_i and $\overline{u_i}$, else $e_i, g_i \notin S$, which in turn would imply that ℓ_i and h_i are not located by S. So S is not an *ILD*-set of G. Similarly, if $\{u_i, \overline{u_i}\} \cap S = \emptyset$, then b_i and d_i are not located by S, again contradicting that S is an *ILD*-set of G. Consequently, either $\{u_i, g_i\} \subseteq S$ or $\{\overline{u_i}, e_i\} \subseteq S$. By independence of S, $\alpha_j \notin S$. Since for all $j \in [m]$, $N(x_j) \cap S \neq N(y_j) \cap S$, α_j is adjacent to one of the literal vertices v say, that belongs to S and corresponds to a literal occurring in clause c_j. Thus setting u_i as True if and only if $u_i \in S$ produces a satisfying truth assignment for ϕ. This completes the proof of the equivalence between (i) and (ii). □

We now use the NP-complete problem P3-SAT$_{\overline{3}}$ [Berman et al. 2003], described below, to prove that ILD is NP-complete, even when restricted to planar triangle-free subcubic graphs (SCILD).

PLANAR 3-SAT WITH AT MOST 3 OCCURRENCES PER VARIABLE (P3-SAT$_{\overline{3}}$) **Instance:** Set U of variables and collection C of clauses over U, $\|U\| = n$ and $\|C\| = m$, such that: (i) each clause $c \in C$ satisfies $\|c\| = 2$ or $\|c\| = 3$; (ii) each variable has two or three occurrences and each negative literal occurs once in C; (iii) the bipartite undirected graph $G = (V, E)$ is planar and connected, where $V = U \cup C$ and E contains edge uc if and only if either u or $\overline{u}$ belongs to clause c. **Question:** Is there a satisfying truth assignment for U satisfying each clause of C?

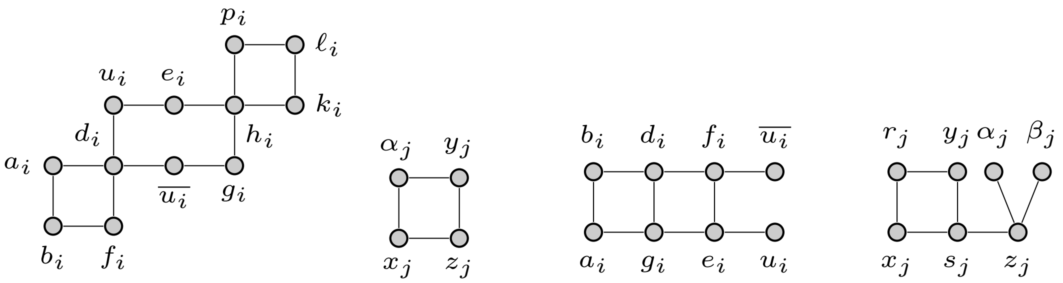
Theorem 2.2. *Deciding, for a given planar triangle-free subcubic graph G, if G has an* ILD*-set is an NP-complete problem.*

Proof. Since the problem is in NP, we prove that P3-SAT$_{\overline{3}}$ is polynomially reducible to the problem SCILD. Let $U = \{u_1, u_2, \ldots, u_n\}$ and $C = \{c_1, c_2, \ldots, c_m\}$ define an instance ϕ of P3-SAT$_{\overline{3}}$. For each variable $u_i \in U$, add a copy of the gadget H_i depicted in Figure 2(c). For each $c_j \in C$, add a copy of the gadget Z_j depicted in Figure 2(d). Let $|c_j| = k$ and $v_{\ell,j} \in \{u_i, \bar{u}_i\}$ be the literals of c_j, for $\ell \in [k]$. For each clause c_j, we add to $E(G)$ the set of edges $\{\alpha_j v_{1,j}, \beta_j v_{2,j} \mid j \in [m]\}$. If $|c_j| = 3$, then we additionally add the set $\{\beta_j v_{3,j} \mid j \in [m]\}$. The resulting graph G has order $|V(G)| = 8n + 7m$ and size at most $|E(G)| = 9n + 10m$, and can be constructed in polynomial time.

The graph G is triangle-free since H_i and Z_j are also triangle-free, and it is not possible to form a triangle containing the vertices of H_i and Z_j. Moreover, G is subcubic since H_i and Z_j are subcubic and each element of $\{u_i, \bar{u}_i\}$ is in at most two clauses. We are now proceed to prove that (i) G admits an *ILD*-set if and only if (ii) ϕ is satisfiable.

To establish that (i) is sufficient for (ii), assume that ϕ admits a satisfying truth assignment. Define $D_1 = \{u_i, b_i, g_i, f_i \mid u_i \text{ is True}\}$ and $D_2 = \{\overline{u_i}, a_i, d_i, e_i \mid u_i \text{ is False}\}$. Let $S = D_1 \cup D_2 \cup \{x_j, y_j, z_j \mid j \in [m]\}$. It follows directly that S is an *ILD*-set of G.

To prove that (ii) is sufficient for (i), suppose that G admits an *ILD*-set S. For each $i \in [n]$, either $\{a_i, d_i, e_i\} \subseteq S$ or $\{b_i, g_i, f_i\} \subseteq S$, which implies that $|S \cap \{u_i, \overline{u_i}\}| \leqslant 1$. Since x_j and y_j are false twins, for each $j \in [m]$, $\{x_j, y_j\} \subseteq S$. Moreover, $N(x_j) \cap S \neq N(y_j) \cap S$, implies that $z_j \in S$ and hence $\{\alpha_j, \beta_j\} \cap S = \emptyset$. If $\{u_i, \overline{u_i}\} \cap S = \emptyset$, since $\{\alpha_j, \beta_j\} \cap S = \emptyset$, one of these two vertices is not dominated by S. Thus, $|S \cap \{u_i, \overline{u_i}\}| = 1$. Since $N(\alpha_j) \cap S \neq N(\beta_j) \cap S$, at least one member of $\{\alpha_j, \beta_j\}$ is adjacent to one of the vertices $v_{\ell,j}$ corresponding to a literal contained in the clause c_j. Thus, setting u_i as True if and only if $u_i \in S$ produces a satisfying truth assignment for ϕ. □



(a) $\mathrm{H_i}$ for variable $\mathrm{u_i}$. (b) $\mathrm{Z_j}$ for clause $\mathrm{c_j}$. (c) $\mathrm{H_i}$ for variable $\mathrm{u_i}$. (d) $\mathrm{Z_j}$ for clause $\mathrm{c_j}$.

Figure 2. Gadgets for reductions in Theorems 2.1, (a) and (b), and 2.2, (c) and (d).

Concluding remarks

We established that the INDEPENDENT LOCATING-DOMINATING SET problem remains NP-complete even when restricted to planar bipartite graphs and planar triangle-free subcubic graphs, underscoring the inherent computational intractability of this problem. Prominent directions for future work include the design of polynomial-time algorithms for suitably restricted and well-structured subclasses of these graph families, as well as the derivation of tight bounds on $ILD(G)$ under additional structural assumptions.

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