On r-dynamic coloring of graphs in subclasses of planar and circulant graphs

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Abstract. *An r-dynamic coloring of a graph G is a proper vertex coloring in which each vertex sees at least $\min\{r, d(v)\}$ distinct colors in its neighborhood. The minimum number of colors in such a coloring is the r-dynamic chromatic number $\chi_r^d(G)$. We determine exact values and upper bounds of χ_r^d for several graph classes, including triangular grids, planar 3-trees for $r \leq 4$, and planar Eulerian triangulations for $r \leq 3$ (with a partial result for $r = 4$), confirming the conjecture of [Song et al. 2014] for these subclasses. We also establish exact values for the 2-dynamic chromatic number of a subclass of circulant graphs, confirming a conjecture of [Montgomery 2001] for this regular family.*

1. Introduction

In this paper we consider only simple (undirected, without loops or parallel edges) graphs, and the terminology and notation are standard [Bondy and Murty 2008, Diestel 2017]. The neighborhood set of a vertex $v \in V(G)$ is denoted by $N(v)$ and the degree of v, which addresses the cardinality of $N(v)$, is denoted by $d(v)$. Set $[k] = \{1, 2, \ldots, k\}$.

An r-dynamic *coloring* of G is a proper vertex coloring $c : V \to S$ such that for every vertex $v \in V$, the number of distinct colors from the set S assigned to its neighborhood $N(v)$ is at least $\min\{r, d(v)\}$. The smallest integer k for which G admits an r-dynamic coloring with k colors is called the *r-dynamic chromatic number* of G and is denoted by $\chi_r^d(G)$. A graph G is (k, r)-*colorable* if it admits an r-dynamic coloring using at most k colors.

The following chain of inequalities was established in 2006 by [Lai et al. 2006].

Proposition 1. $\chi(G) = \chi_1^d(G) \leq \cdots \leq \chi_{\Delta(G)}^d(G)$

In addition, they show that $\min\{r, \Delta(G)\} + 1 \leq \chi_r^d(G) \leq \chi_{L,r}^d(G) \leq |V(G)|$.

For circulant graphs, the following is a result found by [Dafik et al. 2017].

Theorem 2. $\chi_2^d(C_p(1, \frac{p}{2})) = 4$

Regarding near triangulation graphs, [Gu et al. 2021] determined the following.

Theorem 3. $\chi_{L,3}^d(G) \leq 6$

Further results on r-dynamic coloring are compiled in the survey of [Chen et al. 2022]. A core conjecture regarding upper bounds for the r-dynamic chromatic number was proposed by Song et al.

Conjecture 4. *[Song et al. 2014]. If G is a planar graph, then*

$$\chi_r^d(G) \leq \begin{cases} r+3, & \text{if } 1 \leq r \leq 2, \\ r+5, & \text{if } 3 \leq r \leq 7, \\ \left\lfloor \frac{3r}{2} \right\rfloor + 1, & \text{if } r \geq 8. \end{cases}$$

For the regular case, Montgomery conjectured the following

Conjecture 5. *[Montgomery 2001]. For every regular graph G, $\chi_2^d(G) \leq \chi(G)$+2.*

In this paper, we progress towards these conjectures and prove the following.

- Exact values and upper bounds for $\chi_r^d(G)$ when G is a triangular grid, and thus we confirm Song's conjecture for this subclass of planar graphs (Theorem 6). This complements the result of [Gu et al. 2021] by considering r-dynamic colorings of near triangulations for multiple values of r.
- Exact values and upper bounds for $\chi_r^d(G)$ when G is a planar 3-tree and $r \in \{3, 4\}$, and thus we confirm Song's conjecture for this subclass of planar graphs (Theorem 7). Our result is lower than the sharp general bound $\chi_3^d(G) \leq 5$ for planar triangulations proved by [Asayama et al. 2018].
- Exact values and upper bounds for $\chi_r^d(G)$ when G is a planar Eulerian triangulation and $r \in \{2, 3\}$, and thus we confirm Song's conjecture for this subclass of planar graphs (Theorem 8). Our result for $r = 3$ shows that there exist planar Eulerian triangulations with a lower value than the upper bound presented by [Asayama et al. 2018].
- An upper bound for $\chi_6^d(G)$ when G belongs to a particular class of planar Eulerian triangulations, and thus we confirm Song's conjecture for this subclass of planar graphs (Theorem 9).
- Exact values for $\chi_2^d(G)$ when G belongs to a subclass of circulant graphs, thus we confirm Montgomery's conjecture for this class of regular graphs (Theorem 10). Our result is consistent with the theorem of [Dafik et al. 2017] for the family $C_p(1, \frac{p}{2})$ when $p = 6$, since both results yield $\chi_2^d = 4$.

2. Triangular Grids

Given a positive integer n, the triangular grid graph T_n is determined by the coordinates $(i,\ j,\ k)$ in a three-dimensional space such that $i + j + k = n$ and $i,\, j,\, k \geq 0$. The edges of T_n are those between pairs of coordinates with Manhattan distance equal to 2 [West et al. 2001]. Note that $\Delta(T_n) = 6$ when $n \geq 3$. Thus, we analyze χ_r^d when $2 \leq r \leq 6$ and obtain the following result.

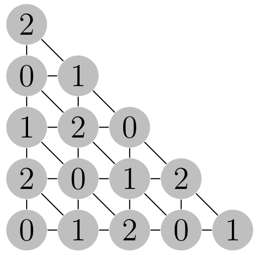
Theorem 6. *For every triangular grid T_n, $\chi_r^d(T_n) \leq r + 2$ if $r \in \{4, 5\}$ and*

$$\chi_r^d(T_n) = \begin{cases} 3, & \text{if } r = 2, \\ 3, & \text{if } r = 3 \text{ and } n = 1, \\ 4, & \text{if } r = 3 \text{ and } n \geq 2, \\ \left|V(T_n)\right|, & \text{if } r = 6 \text{ and } n \leq 2, \\ 7, & \text{if } r = 6 \text{ and } n \geq 3. \end{cases}$$

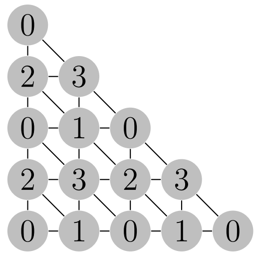
Proof. (Sketch). Due to space constraints, we confine the exposition to the cases $r = 2$ and $r = 3$; the remaining cases are presented in detail in the appendix.

We first establish that $\chi_2^d(T_n) \leq 3$. Define a mapping $\phi\colon V(T_n) \to \{0, 1, 2\}$ by $\phi\big((i, j)\big) = (2j + i) \bmod 3$, as shown in Figure 1a. A direct inspection of the adjacency relations in T_n shows that ϕ is a proper 2-dynamic 3-coloring of T_n, and therefore $\chi_2^d(T_n) \leq 3$.

Next, consider the case $r = 3$ with $n \geq 2$. Define $\phi' \colon V(T_n) \to \{0, 1, 2, 3\}$ by $\phi'\big((i, j)\big) = 2\big(j \bmod 2\big) + \big(i \bmod 2\big)$, as illustrated in Figure 1b. The periodic structure of this assignment guarantees that adjacent vertices are colored differently and that each vertex v has at least $\min\{3, d(v)\}$ distinct colors in its neighborhood. Consequently, ϕ' defines a proper 3-dynamic 4-coloring of T_n, implying $\chi^d_3(T_n) \leq 4$.



(a) The 3-coloring ϕ assigned on T_4



(b) The 4-coloring ϕ' assigned on T_4

Figure 1. Coloring assignments on T_4.

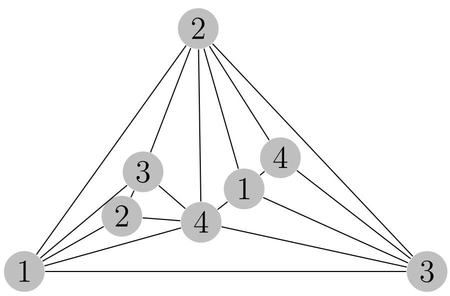
In both cases, the verification relies on the regular combinatorial structure of T_n and the modular nature of the coloring functions. $\square$

3. Planar 3-trees and Planar Eulerian triangulations

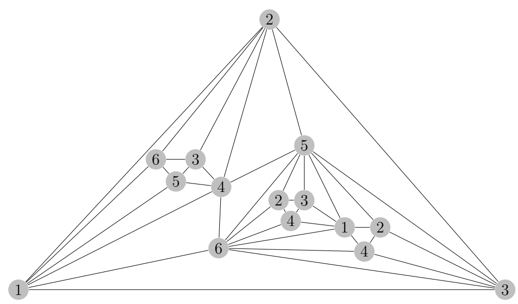
A *planar* 3*-tree* is a planar graph that can be obtained from a triangle by repeatedly choosing one of its current faces and adding a new vertex while joining this new vertex to the three vertices of the face. We show the following.

Theorem 7. *If G is a planar 3-tree with at least 4 vertices then $\chi^d_r(G) = 4$ if $r \leq 3$ and $\chi^d_r(G) \leq 5$ if $r = 4$.*

Proof. (Sketch). To take advantage of the available space, we select cases $r = 2, 3$. This proof consists of induction on the number of vertices of a planar 3-tree G. For the base case K_4, it is known that $\chi(K_4) = \chi^d_2(K_4) = \chi^d_3(K_4) = 4$. Using Proposition 1, this implies that $\chi^d_3(G) \geq \chi^d_2(G) \geq 4$. For a planar 3-tree $G' = G - v$, we assume that it is $(4, 3)$-colorable by a mapping $\phi : V(G') \to [4]$. After adding a new vertex v, it receives the remaining color from $[4] - \phi\big(N(v)\big)$, which leads v to follow the 3-dynamic proper constraint. In the same way, the involved neighbors of v which are a, b and c fulfill those constraints as well. The color distribution is shown in Figure 2a. $\square$



(a) $(4, 3)$-coloring ϕ assigned on a sample planar 3-tree.



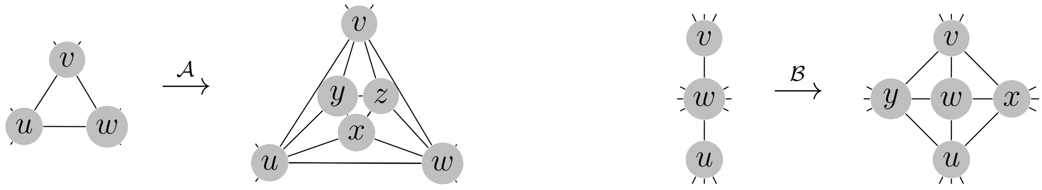
(b) $(6, 4)$-coloring ϕ assigned on a planar Eulerian triangulation generated by $\mathcal{A}$-operations.

Figure 2. Examples of dynamic colorings on planar triangulations.

A *planar Eulerian triangulation* is a planar triangulation in which every vertex has even degree. We show the following.

Theorem 8. *If G is a planar Eulerian triangulation, then $\chi^d_r(G) = 3$ for $r = 2$, and also for $r = 3$ when $|V(G)| \leq 5$. If $r = 3$ and $|V(G)| > 5$, then $4 \leq \chi^d_r(G) \leq 5$. Moreover, both the upper bound and lower bound are tight.*

According to [Matsumoto and Nakamoto 2015], the following operations are enough to generate every planar Eulerian triangulation starting from the octahedron.



(a) Visual representation of operation $\mathcal{A}$ applied to face f.

(b) Visual representation of operation $\mathcal{B}$ applied to vertices u, v, w.

Figure 3. Representation of operations $\mathcal{A}$ and $\mathcal{B}$.

Operation $\mathcal{A}(G, f)$: Inserts a triangle xyz into a face $f = uvw$ in G and adds the edges ux, vy, wz, uy, vz and wx. (Figure 3a).

Operation $\mathcal{B}(G, u, v, w)$: Given a path vwu we insert two vertices $x, y \notin V(G)$ and the set of edges $\{yw, xw, vy, vx, uy, ux\}$. Then, we remove every neighbor of w distinct from u and v, and distribute the neighbors into neighbors of x and y. (Figure 3b).

We present the following result for this class.

Theorem 9. *If G is a planar Eulerian triangulation that can be obtained from the octahedron by a sequence of applications of the operation $\mathcal{A}$, then $\chi^d_4(G) \leq 6$. (Figure 2b).*

4. Circulant graphs

Let $p \in \mathbb{N}$ and let $n_1, n_2, \ldots, n_k$ be integers such that $0 < n_1 < n_2 < \cdots < n_k < \frac{p+1}{2}$. The *circulant graph* $C_p(n_1, n_2, \ldots, n_k)$ is the graph with vertex set $\{0, 1, \ldots, p-1\}$ and edge set $E(G) = \{(i, j) : j \equiv i \pm n_t \pmod{p},\ 1 \leq t \leq k\}$.

Theorem 10. *Let $p \geq 6$. Then* $\chi^d_2(C_p(1,3)) = \begin{cases} 3, & \text{if } p \equiv 0 \pmod{9}, \\ 4, & \text{otherwise.} \end{cases}$

Proof. We confine the exposition to the case when $p \equiv 0 \pmod{4}$. Define a coloring $\phi : V(C_p(1,3)) \to \{0,1,2,3\}$ by $\phi(i) = i \pmod{4}$. Since $p \equiv 0 \pmod{4}$, ϕ is proper. For any vertex i, the neighbors $i \pm 1$ and $i \pm 3$ receive the colors $i + 1 \pmod{4}$ and $i + 3 \pmod{4}$, which are distinct. Thus, ϕ is 2-dynamic and $\chi^d_2(C_p(1,3)) \leq 4$ (Figure 4). $\square$

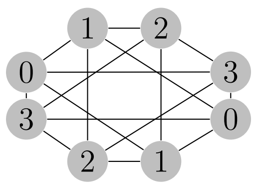


Figure 4. Case $p \equiv 0 \pmod{4}$, assignment of coloring ϕ on graph $C_8(1,3)$.

Corollary 11. *For every $(1,3)$-circulant graph G, we have $\chi^d_2(G) \leq \chi(G) + 2$.*

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Appendix

We shall repeatedly invoke the following proposition, which provides a general lower bound for $\chi_r^d(G)$.

Proposition 12. *[Lai et al. 2006]. For any graph G,* $\min\{r, \Delta(G)\} + 1 \leq \chi_r^d(G) \leq |V(G)|$.

A. Proofs for Triangular Grids

The next observation is clear.

Observation 13. *If $(i,j) \in V(T_n)$, then* $N\big((i,j)\big) \subseteq \big\{(i-1,j),(i+1,j),(i,j-1),(i,j+1),(i+1,j-1),(i-1,j+1)\big\}$ *and* $d\big((i,j)\big) \in \{2,4,6\}$.

A.1. Proof for $r = 2$

Our aim is to prove $\chi_2^d(T_n) \leq 3$. We define a coloring $\phi : V(T_n) \to \{0,1,2\}$ by the next rule

$$\phi\big((i,j)\big) = (2j+i) \bmod 3.$$

We now prove that ϕ is a proper $(3,2)$-coloring of T_n.

First, we verify that ϕ is proper. By Observation 13,

$$\begin{aligned}\phi\big(N\big((i,j)\big)\big) \subseteq \big\{&(2j+i-1) \bmod 3,\ (2j+i+1) \bmod 3,\ (2j+i-2) \bmod 3,\\ &(2j+i+2) \bmod 3,\ (2j+i-1) \bmod 3,\ (2j+i+1) \bmod 3\big\}.\end{aligned}$$

Reducing modulo 3, this yields

$$\phi\big(N\big((i,j)\big)\big) \subseteq \big\{(2j+i+1) \bmod 3,\ (2j+i+2) \bmod 3\big\}.$$

Since $\phi\big((i,j)\big) = (2j+i) \bmod 3$ is distinct from both $(2j+i+1) \bmod 3$ and $(2j+i+2) \bmod 3$, it follows that $\phi\big((i,j)\big) \notin \phi\big(N\big((i,j)\big)\big)$ for every vertex (i,j). Hence, ϕ is a proper 3-coloring as seen in Figure 5.

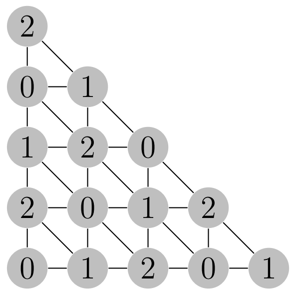


Figure 5. The 3-coloring ϕ assigned on T_4

Next, we verify the 2-dynamic condition. By Observation 13, $\delta(T_n) \geq 2$, so it suffices to show that $\big|\phi\big(N\big((i,j)\big)\big)\big| \geq 2$ for every $(i,j) \in V(T_n)$. Since the previous inclusion shows that $\phi\big(N\big((i,j)\big)\big) \subseteq \{(2j+i+1) \bmod 3, (2j+i+2) \bmod 3\}$, it is enough to prove that both of these colors actually occur among the neighbors of (i,j).

Case 1. $d\big((i,j)\big) = 2$. In this case, $(i,j) \in \{(0,0),(0,n),(n,0)\}$. By the definition of T_n, if $N\big((i,j)\big) = \{(a,b),(e,f)\}$ then $(a,b)(e,f) \in E(T_n)$, so the two neighbors are adjacent. Since ϕ is proper, adjacent vertices receive distinct colors, and therefore $\big|\phi\big(N\big((i,j)\big)\big)\big| = 2$.

Case 2. $d\big((i,j)\big) = 4$. In this case, we have that

$$\begin{aligned} N\big((i,j)\big) \in \big\{ &\{(i,j-1),(i,j+1),(i+1,j-1),(i+1,j)\}, \\ &\{(i-1,j),(i+1,j),(i-1,j+1),(i,j+1)\}, \\ &\{(i-1,j+1),(i+1,j-1),(i-1,j),(i,j-1)\}\big\}. \end{aligned}$$

If $N\big((i,j)\big) = \{(i,j-1),(i,j+1),(i+1,j-1),(i+1,j)\}$ or $N\big((i,j)\big) = \{(i-1,j),(i+1,j),(i-1,j+1),(i,j+1)\}$, then both $(i+1,j)$ and $(i,j+1)$ belong to $N\big((i,j)\big)$, and

$$\phi\big((i+1,j)\big) = (2j+i+1) \bmod 3, \qquad \phi\big((i,j+1)\big) = (2j+i+2) \bmod 3.$$

If instead $N\big((i,j)\big) = \{(i-1,j+1),(i+1,j-1),(i-1,j),(i,j-1)\}$, then $(i,j-1)$ and $(i-1,j)$ belong to $N\big((i,j)\big)$, and

$$\phi\big((i,j-1)\big) = (2j+i-2) \bmod 3 = (2j+i+1) \bmod 3, \qquad \phi\big((i-1,j)\big) = (2j+i-1) \bmod 3 = (2j+i+2)$$

In all cases, both colors occur in the neighborhood, and thus $\big|\phi\big(N\big((i,j)\big)\big)\big| = 2$.

Case 3. $d\big((i,j)\big) = 6$. In this case, we have that

$$N\big((i,j)\big) = \big\{(i-1,j),(i+1,j),(i,j-1),(i,j+1),(i+1,j-1),(i-1,j+1)\big\}.$$

In particular, $(i+1,j) \in N\big((i,j)\big)$ and $(i,j+1) \in N\big((i,j)\big)$, and as above

$$\phi\big((i+1,j)\big) = (2j+i+1) \bmod 3, \qquad \phi\big((i,j+1)\big) = (2j+i+2) \bmod 3.$$

Hence, $\big|\phi\big(N\big((i,j)\big)\big)\big| = 2$. This is because the colors of $(i+1,j)$ and $(i,j+1)$ are distinct and are the only two colors in the set $\{(2j+i+1) \bmod 3, (2j+i+2) \bmod 3\}$.

Combining the three cases, we conclude that $\big|\phi\big(N\big((i,j)\big)\big)\big| = 2$ for all $(i,j) \in V(T_n)$. Therefore, ϕ is a 2-dynamic 3-coloring of T_n, and thus $\chi_2^d(T_n) \le 3$.

By Proposition 12, $\chi_r^d(G) \ge \min\{r, \Delta(G)\} + 1$. Since $\Delta(T_n) \in \{2,4,6\}$, when $r = 2$ we have $\chi_2^d(T_n) \ge 3$. Therefore, $\chi_2^d(T_n) = 3$ for all $n \ge 1$.

A.2. Proof for $r = 3$

We want to prove $\chi_3^d(T_n) \le 3$ if $n = 1$ and $\chi_3^d(T_n) \le 4$ if $n > 1$. If $n = 1$, Proposition 12 yields $\chi_3^d(T_1) = 3$. Assume $n \ge 2$. Define $\phi : V(T_n) \to \{0,1,2,3\}$ by

$$\phi(i,j) = 2\big(j \bmod 2\big) + \big(i \bmod 2\big).$$

It is shown below that ϕ is a $(4,3)$-coloring. Fix $(i,j) \in V(T_n)$. By Observation 13,

$$\begin{aligned} \phi\big(N\big((i,j)\big)\big) \subseteq \big\{ &2\big(j \bmod 2\big) + (i-1) \bmod 2,\ 2\big(j \bmod 2\big) + (i+1) \bmod 2, \\ &2\big((j-1) \bmod 2\big) + i \bmod 2,\ 2\big((j+1) \bmod 2\big) + i \bmod 2, \\ &2\big((j-1) \bmod 2\big) + (i+1) \bmod 2,\ 2\big((j+1) \bmod 2\big) + (i-1) \bmod 2\big\}. \end{aligned}$$

After reduction modulo 2, it follows that

$$\begin{aligned}\phi\big(N\big((i,j)\big)\big) \subseteq \big\{&2\big(j \bmod 2\big) + (i+1) \bmod 2,\\ &2\big((j+1) \bmod 2\big) + i \bmod 2,\\ &2\big((j+1) \bmod 2\big) + (i+1) \bmod 2\big\}.\end{aligned}$$

In particular, $\phi\big((i,j)\big) \notin \phi\big(N\big((i,j)\big)\big)$, and hence ϕ is a proper coloring as seen in Figure 6.

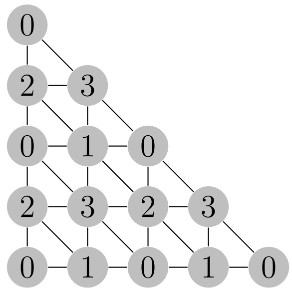


Figure 6. The 4-coloring ϕ assigned on T_4

We are left with verifying that ϕ is 3-dynamic. That is, for every $(i,j) \in V(T_n)$,

$$\big|\phi\big(N\big((i,j)\big)\big)\big| \geq \min\{d\big((i,j)\big), 3\}.$$

By Observation 13, there are three degree cases.

Case 1. $d\big((i,j)\big) = 2$. Here $(i,j) \in \{(0,0),(0,n),(n,0)\}$. By the definition of T_n, if $N\big((i,j)\big) = \{(a,b),(e,f)\}$ then $(a,b)(e,f) \in E(T_n)$, so the two neighbors are adjacent. Since ϕ is proper, these neighbors receive distinct colors, and thus $\big|\phi\big(N\big((i,j)\big)\big)\big| = 2$.

Case 2. $d\big((i,j)\big) = 4$. In this case, it suffices to show that $\big|\phi\big(N\big((i,j)\big)\big)\big| \geq 3$. Moreover,

$$\begin{aligned}N\big((i,j)\big) \in \big\{&\{(i,j-1),(i,j+1),(i+1,j-1),(i+1,j)\},\\ &\{(i-1,j),(i+1,j),(i-1,j+1),(i,j+1)\},\\ &\{(i-1,j+1),(i+1,j-1),(i-1,j),(i,j-1)\}\big\}.\end{aligned}$$

Consequently,

$$\begin{aligned}\phi\big(N\big((i,j)\big)\big) \in \big\{&\{2\big((j+1) \bmod 2\big) + i \bmod 2,\\ &2\big((j+1) \bmod 2\big) + (i+1) \bmod 2,\ 2\big(j \bmod 2\big) + (i+1) \bmod 2\},\\ &\{2\big(j \bmod 2\big) + (i+1) \bmod 2,\\ &2\big((j+1) \bmod 2\big) + (i+1) \bmod 2,\ 2\big((j+1) \bmod 2\big) + i \bmod 2\},\\ &\{2\big((j+1) \bmod 2\big) + (i+1) \bmod 2,\\ &2\big(j \bmod 2\big) + (i+1) \bmod 2,\ 2\big((j+1) \bmod 2\big) + i \bmod 2\}\big\}.\end{aligned}$$

Each set displayed above contains three distinct values, and therefore $\left|\phi\big(N\big((i,j)\big)\big)\right| = 3$.

Case 3. $d\big((i,j)\big) = 6$. In this case, it suffices to show that $\left|\phi\big(N\big((i,j)\big)\big)\right| \geq 3$. Here

$$N\big((i,j)\big) = \big\{(i-1,j),(i+1,j),(i,j-1),(i,j+1),(i+1,j-1),(i-1,j+1)\big\}.$$

Hence,

$$\begin{aligned}\phi\big(N\big((i,j)\big)\big) = \big\{&2\big(j \bmod 2\big) + (i+1) \bmod 2,\\ &2\big((j+1) \bmod 2\big) + i \bmod 2,\\ &2\big((j+1) \bmod 2\big) + (i+1) \bmod 2\big\}.\end{aligned}$$

Consequently, $\left|\phi\big(N\big((i,j)\big)\big)\right| = 3$.

A.3. Proof for $r = 6$

We first address separately the cases $n \in \{1,2\}$.

For $n = 1$, Proposition 12 yields $\chi^d_6(T_1) = 3$.

For $n = 2$, Proposition 12 gives $5 \leq \chi^d_6(T_2) \leq 6$. Nevertheless, for any proper 5-coloring of T_2 there exists a vertex $v \in V(T_2)$ such that $|\phi(N(v))| < d(v)$. It follows that $\chi^d_6(T_2) = 6$.

Assume now that $n \geq 3$. For each $(i,j) \in V(T_n)$, define $\phi(i,j) = (i+5j) \bmod 7$.

We claim that ϕ is a $(7,6)$-coloring. Fix $(i,j) \in V(T_n)$. By Observation 13,

$$\begin{aligned}\phi\big(N\big((i,j)\big)\big) \subseteq \big\{&(i+5j-1) \bmod 7, (i+5j+1) \bmod 7, (i+5j-5) \bmod 7,\\ &(i+5j+5) \bmod 7, (i+5j-4) \bmod 7, (i+5j+4) \bmod 7\big\}.\end{aligned}$$

Using modular arithmetic, this can be rewritten as

$$\begin{aligned}\phi\big(N\big((i,j)\big)\big) \subseteq \big\{&(i+5j+1) \bmod 7, (i+5j+2) \bmod 7, (i+5j+3) \bmod 7,\\ &(i+5j+4) \bmod 7, (i+5j+5) \bmod 7, (i+5j+6) \bmod 7\big\}.\end{aligned}$$

As shown, $\phi((i,j)) \notin \phi(N(i,j))$, hence ϕ is proper as seen in Figure 7.

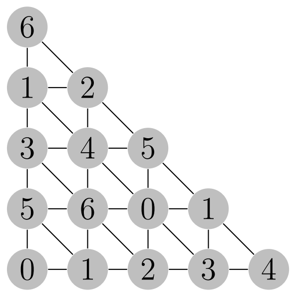


Figure 7. The 7-coloring ϕ assigned on T_4

We proceed to verify that $\left|\phi\big(N\big((i,j)\big)\big)\right| \geq \min\{d\big((i,j)\big), 6\}$ for all $(i,j) \in V(T_n)$. By Observation 13, there are three cases.

Case 1. $d\big((i,j)\big) = 2$. In this case $(i,j) \in \{(0,0),(0,n),(n,0)\}$. By the definition of T_n, if $N\big((i,j)\big) = \{(a,b),(e,f)\}$ then $(a,b)\,(e,f) \in E(T_n)$. Since ϕ is proper, we obtain $\big|\phi\big(N\big((i,j)\big)\big)\big| = 2$.

Case 2. $d\big((i,j)\big) = 4$. In this case, it suffices to show that $\big|\phi\big(N\big((i,j)\big)\big)\big| = 4$. One has

$$\begin{aligned} N\big((i,j)\big) \in \big\{ &\{(i,j-1),(i,j+1),(i+1,j-1),(i+1,j)\},\\ &\{(i-1,j),(i+1,j),(i-1,j+1),(i,j+1)\},\\ &\{(i-1,j+1),(i+1,j-1),(i-1,j),(i,j-1)\}\big\}. \end{aligned}$$

Consequently,

$$\begin{aligned} \phi\big(N\big((i,j)\big)\big) \in \big\{ &\{(i+5j+2) \bmod 7, (i+5j+5) \bmod 7,\\ &(i+5j+3) \bmod 7, (i+5j+1) \bmod 7\},\\ &\{(i+5j+6) \bmod 7, (i+5j+1) \bmod 7,\\ &(i+5j+4) \bmod 7, (i+5j+5) \bmod 7\},\\ &\{(i+5j+4) \bmod 7, (i+5j+3) \bmod 7,\\ &(i+5j+6) \bmod 7, (i+5j+2) \bmod 7\}\big\}. \end{aligned}$$

Thus, $\big|\phi\big(N\big((i,j)\big)\big)\big| = 4$.

Case 3. $d\big((i,j)\big) = 6$. In this case, it remains to verify that $\big|\phi\big(N\big((i,j)\big)\big)\big| = 6$. One has

$$N\big((i,j)\big) = \big\{(i-1,j),(i+1,j),(i,j-1),(i,j+1),(i+1,j-1),(i-1,j+1)\big\}.$$

Consequently,

$$\begin{aligned} \phi\big(N\big((i,j)\big)\big) \subseteq \big\{&(i+5j+1) \bmod 7, (i+5j+2) \bmod 7, (i+5j+3) \bmod 7,\\ &(i+5j+4) \bmod 7, (i+5j+5) \bmod 7, (i+5j+6) \bmod 7\big\}. \end{aligned}$$

Thus, $\big|\phi\big(N\big((i,j)\big)\big)\big| = 6$.

A.4. Proof for $r = 4$

The cases $n < 3$ are treated separately.

For $n = 1$, a 3-coloring of T_1 is given in Figure 8. For each $v \in V(T_1)$, Table 1 shows that the coloring is proper (since $\phi(v) \notin \phi(N(v))$) and 4-dynamic (since $|\phi(N(v))| \geq \min\{4, d(v)\}$). Consequently, $\chi_4^d(T_1) \leq 6$.

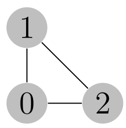


Figure 8. Manual color assignment to T_1.

v	$d(v)$	$\phi(v)$	$\phi(N(v))$
$(0,0)$	2	0	$\{1,2\}$
$(0,1)$	2	1	$\{0,2\}$
$(1,0)$	2	2	$\{0,1\}$

Table 1. Manual coloring of T_1.

For $n = 2$, a 6-coloring of T_2 is given in Figure 9. For each $v \in V(T_2)$, Table 2 shows that the coloring is proper (since $\phi(v) \notin \phi(N(v))$) and 4-dynamic (since $|\phi(N(v))| \geq \min\{4, d(v)\}$). Consequently, $\chi_4^d(T_2) \leq 6$.

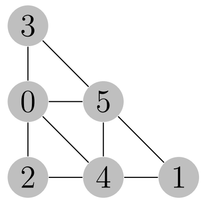


Figure 9. Manual color assignment to T_2.

v	$d(v)$	$\phi(v)$	$\phi(N(v))$
$(0,0)$	2	2	$\{0,4\}$
$(0,1)$	4	0	$\{2,3,4,5\}$
$(1,0)$	4	4	$\{0,1,2,5\}$
$(0,2)$	2	3	$\{0,5\}$
$(1,1)$	4	5	$\{0,1,3,4\}$
$(2,0)$	2	1	$\{4,5\}$

Table 2. Manual coloring of T_2.

Assume $n \geq 3$. For each $(i,j) \in V(T_n)$, define

$$\phi(i,j) = \begin{cases} i \bmod 6, & \text{if } j = 0, \\ (4-j) \bmod 6, & \text{if } i = 0 \text{ and } j > 0, \\ (5+i-j) \bmod 6, & \text{otherwise.} \end{cases}$$

The proof proceeds by induction on n to show that ϕ is a proper $(6,4)$-coloring of T_n.

If $n = 3$, then Table 3 shows that the coloring is proper for all $v \in V(T_3)$ (since $\phi(v) \notin \phi(N(v))$) and 4-dynamic (since $|\phi(N(v))| \geq \min\{4, d(v)\}$). Hence the claim holds for $n = 3$.

v	$d(v)$	$\phi(v)$	$\phi(N(v))$
(0,0)	2	0	{1,3}
(1,0)	4	1	{0,2,3,5}
(0,1)	4	3	{0,1,2,5}
(2,0)	4	2	{0,1,3,5}
(1,1)	6	5	{0,1,2,3,4}
(0,2)	4	2	{1,3,4,5}
(3,0)	2	3	{1,2}
(2,1)	4	0	{2,3,4,5}
(1,2)	4	4	{0,1,2,5}
(0,3)	2	1	{2,4}

Table 3. Coloring ϕ assigned to T_3.

Note that $V(T_n) = V(T_{n-1}) \cup W_n$ and $E(T_n) = E(T_{n-1}) \cup X_n$, where

$$\begin{aligned} W_n =& \{(i, n-i) \mid 0 \leq i \leq n\} \\ X_n =& \{(i, n-i)(i+1, n-i-1) \mid 0 \leq i < n\} \cup \\ & \{(i, n-i-1)(i, n-i) \mid 0 \leq i < n\} \cup \\ & \{(i, n-i)(i+1, n-i) \mid 0 \leq i < n\} \quad . \end{aligned}$$

Now assume $n > 3$, and consider an arbitrary vertex $v \in V(T_n)$. If $v \notin W_{n-1} \cup W_n$, then the neighborhood of v in T_n coincides with its neighborhood in T_{n-1}, and therefore $|\phi(N(v))| \geq \min\{4, d(v)\}$. It remains to consider $v \in W_{n-1} \cup W_n$.

v	$d(v)$	$\phi(v)$	$\phi(N(v))$
$(n, 0)$	2	$n \bmod 6$	$\{(n+3) \bmod 6, (n+5) \bmod 6\}$
$(n-1, 1)$	4	$(n+3) \bmod 6$	$\{n \bmod 6, (n+1) \bmod 6,$ $(n+2) \bmod 6, (n+5) \bmod 6\}$
$(n-k, k)$ for $1 < k < n-1$	4	$(n-2k+5) \bmod 6$	$\{(n-2k) \bmod 6,$ $(n-2k+1) \bmod 6,$ $(n-2k+3) \bmod 6,$ $(n-2k+4) \bmod 6\}$
$(1, n-1)$	4	$(1-n) \bmod 6$	$\{(4-n) \bmod 6, (5-n) \bmod 6,$ $(2-n) \bmod 6, (3-n) \bmod 6\}$
$(0, n)$	2	$(4-n) \bmod 6$	$\{(1-n) \bmod 6, (5-n) \bmod 6\}$
$(n-1, 0)$	4	$(n+5) \bmod 6$	$\{n \bmod 6, (n+2) \bmod 6,$ $(n+3) \bmod 6, (n+4) \bmod 6\}$
$(n-2, 1)$	6	$(n+2) \bmod 6$	$\{n \bmod 6, (n+1) \bmod 6,$ $(n+3) \bmod 6, (n+4) \bmod 6,$ $(n+5) \bmod 6\}$
$(n-k-1, k)$ for $1 < k < n-2$	6	$(n-2k+4) \bmod 6$	$\{(n-2k) \bmod 6,$ $(n-2k+2) \bmod 6,$ $(n-2k+3) \bmod 6,$ $(n-2k+5) \bmod 6\}$
$(1, n-2)$	6	$(2-n) \bmod 6$	$\{(1-n) \bmod 6, (2-n) \bmod 6,$ $(3-n) \bmod 6, (4-n) \bmod 6,$ $(5-n) \bmod 6\}$
$(0, n-1)$	4	$(5-n) \bmod 6$	$\{(4-n) \bmod 6, (2-n) \bmod 6,$ $(1-n) \bmod 6, n \bmod 6\}$

Table 4. Coloring ϕ assigned to $v \in W_n \cup W_{n-1}$.

According to Table 4, the coloring is proper for each listed vertex (since $\phi(v) \notin \phi(N(v))$) and satisfies the 4-dynamic requirement (since $|\phi(N(v))| \geq \min\{4, d(v)\}$). This completes the inductive step and shows that $\chi_4^d(T_n) \leq 6$.

A.5. Proof for $r = 5$

By Proposition 12, we have $\chi_5^d(T_n) \geq 6$. Moreover, the proof in Subsection A.3 together with Proposition 1 imply that $\chi_5^d(T_n) \leq 7$.

B. Proofs for Planar 3-trees

Given a vertex u in a planar 3-tree G, we say that u is the *representative* of a K_4 $\{a, b, c, u\}$ if u was added inside the triangle abc in the construction of G. We denote the corresponding K_4 as $K(u)$.

B.1. Proof for $r = 2$ and $r = 3$

We prove by induction on $|V(G)|$ that every planar 3-tree G with at least 4 vertices admits a (4,3)-coloring. For the base case, we note that $G \simeq K_4$. Using Proposition 12, we observe that $4 \leq \chi_3^d(K_4) \leq 4$. Thus, $\chi_3^d(K_4) = 4$. Otherwise, let v be a vertex of degree 3 in $V(G)$, define $G' = G - v$. We assume that G' is $(4, 3)$-colorable by a mapping

$\phi : V(H') \to [4]$. Once vertex v is inserted, it shall take the remaining color from $[4] \setminus \phi(N(v))$. We observe that v has indeed three distinct colors in its neighborhood. Since v has degree 3, exactly one color in $[4]$ is unused in its neighbors, which can be assigned to v. For vertices $\{a, b, c\}$ which are children of v, we observe that these will also have at least three distinct colors. Which in turn confirms that a $(4, 3)$-coloring is assignable. We conclude that $\chi^d_3(G) = 4$, and consequently $\chi^d_2(G) = 4$. Figure 10 shows an assignment on a sample planar 3-tree.

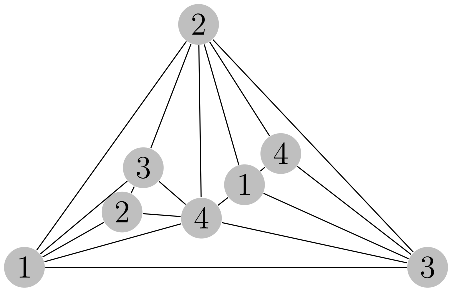
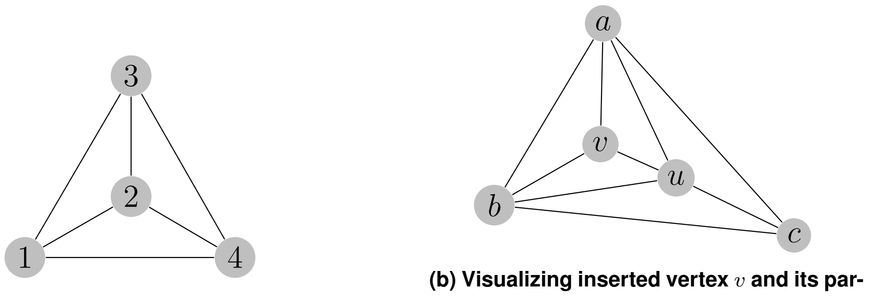


Figure 10. (4,3)-coloring ϕ assigned on a sample planar 3-tree.

B.2. Proof for $r = 4$

We prove by induction on $|V(G)|$ that every planar 3-tree G with at least 4 vertices admits a (5,4)-coloring. For the base case, we note that $G \simeq K_4$. Using Proposition 12, we observe that $4 \leq \chi^d_4(K_4) \leq 4$.



(a) Assigning 4 colors to K_4.

(b) Visualizing inserted vertex v and its parent u.

Figure 11

Let G be a planar 3-tree and v be a vertex of degree 3 in $V(G)$, we define $G' = G - v$. Assume by induction hypothesis that there exists a proper 4-dynamic 5-coloring $\phi : V(H') \to [5]$. Let u be the parent of v in the construction of G, and let $K(u) = \{a, b, c, u\}$. See Figure 11b. Since $K(u)$ induces a K_4 in G', the vertices in $K(u)$ are pairwise adjacent whenever required, and thus they receive pairwise distinct colors under ϕ. Define $\phi(v)$ to be the unique color in $[5] \setminus \phi(K(u))$, where $\phi(K(u)) = \{\phi(x) : x \in K(u)\}$. This extends ϕ to a proper coloring of G.

We now shall verify that the resulting coloring is 4-dynamic. Only the vertices $v, u, a, b,$ require attention, as other vertices do not change their neighborhood. First, for the new vertex v, its three neighbors a, b, u receive pairwise distinct colors. Since $d(v) = 3$, the vertex v sees $\min\{4, d(v)\} = 3$ distinct colors in its neighborhood. Second, consider u. The vertices in $V(K(u) - u) = \{a, b, c\}$ are pairwise adjacent and therefore receive pairwise distinct colors. Moreover, by construction $\phi(v) \notin \phi(K(u))$, so $\phi(v)$ is different from the colors on a, b, c, and u. Hence, the neighborhood of u contains at least four distinct colors.

Third, consider a and b in $V\big(K(v)\big) \setminus \{u\}$. Both vertices have degree at least 4. Among their neighbors are v, u, the other vertex in $\{a, b\}$, and c; these four vertices have been assigned pairwise distinct colors. Therefore, both a and b see at least four distinct colors in their neighborhoods. Consequently, the extended ϕ is a proper 4-dynamic 5-coloring of G. Hence, $\chi_4^d(H) \leq 5$. Figure 12 shows an asigment of this coloring on a sample planar 3-tree.

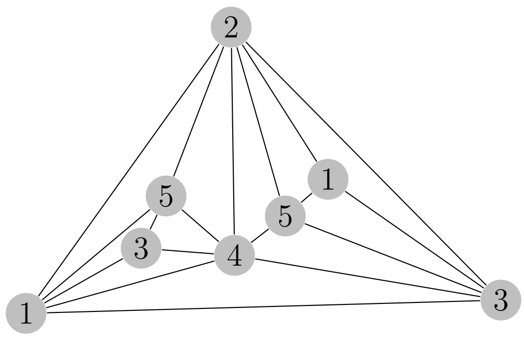


Figure 12. (5,4)-coloring ϕ assigned on a sample planar 3-tree.

C. Proofs for Planar Eulerian Triangulations

Considering the cycle C_n, the double wheel graphDW_n is defined by

$$V(DW_n) = V(C_n) \cup \{u, v\}$$

$$E(DW_n) = E(C_n) \cup \big(\{u, v\} \times V(C_n)\big)$$

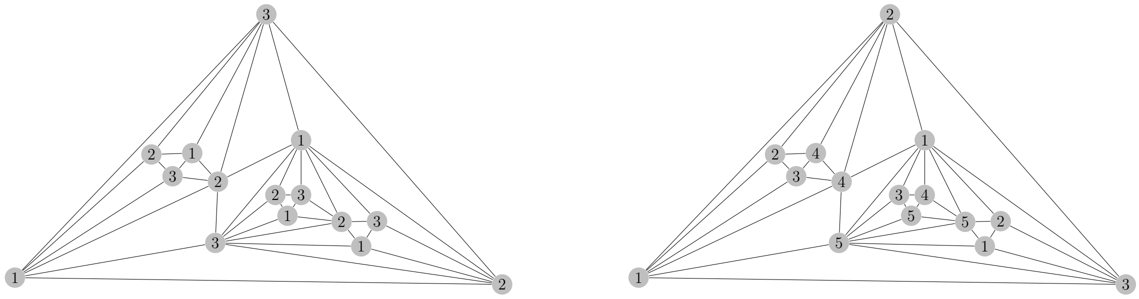
Note that this graph class qualifies as a Eulerian triangulation for $n \equiv 0 \mod 2$.

C.1. Proof for $r = 2$

Recall that we want to prove that for every planar Eulerian triangulation G, it holds that $\chi_2^d(G) = 3$. By [Tsai and West 2011], it is known that every Eulerian planar triangulation is 3-colorable; that is, $\chi(G) = 3$. Therefore, fix a proper 3-coloring $\phi : V(G) \to [3]$. Due to G being Eulerian, for every $v \in V(G)$, its neighborhood shall be even. In particular, $|N(v)| \equiv 0 \pmod 2$, which implies that $|\phi(N(v))| \geq 2$.

Under the 3-coloring ϕ, if the neighbors of vertex v were to receive the same color, two vertices in an induced triangle have the same color which contradicts properness. Thus, the neighborhood of every vertex has two distinct colors. In consequence, ϕ also satisfies the 2-dynamimc condition and is considered as a (3,2)-coloring.

Figure 13a shows a $(3, 2)$-coloring assigned on a planar Eulerian triangulation.



(a) $(3,2)$**-coloring** **(b)** $(5,3)$**-coloring**

Figure 13. Dynamic colorings of a planar Eulerian triangulation.

C.2. Proof for $r = 3$

Proof. As shown in [Asayama et al. 2018], for planar triangulations the bound $\chi_3^d(G) \leq 5$ is tight.

Let G be a planar Eulerian triangulation. If $|V(G)| \leq 5$, then G is a planar graph isomorphic to K_3. By Proposition 12, it follows that $\chi_3^d(G) = 3$.

Now suppose that $|V(G)| > 5$. Since G is Eulerian, every vertex has even degree, and for planar triangulations we have $\delta(G) = 4$. Hence $\Delta(G) \geq \delta(G) = 4$. By Proposition 12, this implies that $\chi_3^d(G) \geq 4$.

We next show that the lower bound 4 is attainable, and that the upper bound 5 is also tight.

Proposition 14. *If* $n \equiv 0 \pmod 3$*, then* $\chi_3^d(DW_n) = 4$.

Proof. From Theorem 2.5 in [Lai et al. 2006], we know that $\chi_3^d(C_n) = 3$ whenever $n \equiv 0 \pmod 3$.

Let $C_n \subset DW_n$, and denote by x and y the two vertices of DW_n that are not in $V(C_n)$. Fix a $(3,3)$-coloring ϕ' of the induced subgraph C_n. For every vertex $z \in V(C_n)$, we then have

$$|\phi'(N(z))| = 2,$$

since each vertex of the cycle has exactly two neighbors in C_n.

When the vertices x and y are added to form DW_n, each vertex $z \in V(C_n)$ gains two additional neighbors, namely x and y, and therefore has four neighbors in DW_n.

Define a coloring ϕ of DW_n by setting $\phi(z) = \phi'(z)$ for all $z \in V(C_n)$. To ensure that ϕ is 3-dynamic, the vertices x and y must receive a color not used in $\phi'(V(C_n))$. Let k^* be a new color, and assign

$$\phi(x) = \phi(y) = k^*.$$

Under this assignment, each vertex $z \in V(C_n)$ has neighbors colored with the two colors appearing in $\phi'(N(z))$ together with the color k^*, giving exactly three distinct colors in $N(z)$. Moreover, the neighborhoods of x and y consist precisely of the vertices of C_n, whose colors are the three colors used in ϕ'. Hence $|\,\phi(N(x))\,| = |\,\phi(N(y))\,| = 3$.

Therefore ϕ is a $(4,3)$-coloring of DW_n, and consequently $\chi_3^d(DW_n) \leq 4$. Since $\chi_3^d(DW_n) \geq 4$ by Proposition 12, we conclude that $\chi_3^d(DW_n) = 4$.

An example of such a coloring for a sample graph DW_n is shown in Figure 14a. □

By Proposition 14, the lower bound $\chi_3^d(G) \geq 4$ is tight. Furthermore, the coloring of the double wheel graph DW_4 shown in Figure 14b demonstrates that the upper bound $\chi_3^d(G) \leq 5$ is also tight.

(a) $(4,3)$**-coloring assigned to** DW_6**.** **(b)** $(5,3)$**-coloring assigned to** DW_4**.**

Figure 14. 3**-dynamic colorings assigned to double wheel graphs.**

In Figure 13b, it is illustrate a $(5,3)$-coloring of a planar Eulerian triangulation obtained by brute-force search. Observe that the neighborhood of every vertex contains three distinct colors, satisfying the 3-dynamic coloring condition. □

C.3. Proof of Theorem 9

Recall that we want to prove that If G is a planar Eulerian triangulation that can be obtained from the octahedron by a sequence of applications of the operation $\mathcal{A}$, then $\chi_4^d(G) \leq 6$.

Proposition 15. *For every n,*

$$\chi_4^d(DW_n) \leq \begin{cases} 7, & \textit{if } n = 5, \\ 6, & \textit{otherwise}. \end{cases}$$

Proof. We invoke Theorem 2.5 from [Lai et al. 2006] which holds that for every r and n,

$$\chi_r^d(C_n) = \begin{cases} 5, & \text{if } n = 5, \\ 3, & \text{if } n \equiv 0 \pmod 3, \\ 4, & \text{otherwise}. \end{cases}$$

Let $C_n \subset DW_n$ and let vertices $x, y \notin V(C_n)$. Fix a $(\chi_4^d(C_n), 4)$-coloring ϕ' of the subgraph induced by C_n. Upon adding x and y, every vertex of C_n has four neighbors in DW_n. Let ϕ be a coloring of DW_n such that $\phi(v) = \phi'(v)$ for all $v \in C_n$. By the definition of a 4-dynamic coloring, we must have $\phi(x) \neq \phi(y)$ and, moreover, $\phi(x), \phi(y) \notin \phi(V(C_n))$.

Consequently, when $n \not\equiv 0 \pmod 3$, the extension from C_n to DW_n introduces two additional colors to be assigned to x and y respectively. Therefore, $\chi_4^d(DW_n) \leq \chi_4^d(C_n) + 2$.

Similarly, when $n \equiv 0 \pmod 3$, after extending to ϕ following the same procedure as the previous case, the vertices x and y each see only three distinct colors in their neighborhoods. Hence, one more color k^* is added to the codomain of ϕ, which is used to recolor a single vertex $v \in C_n$ by setting $\phi(v) = k^*$. It follows that $\chi_4^d(DW_{0 \bmod 3}) = \chi_4^d(C_{0 \bmod 3}) + 3 \leq 6$. □

Let $S = [6]$, and let $\phi : V(G) \to S$ be a $(6,4)$-coloring. Let G_0 denote the octahedral graph which is isomorphic to DW_4. By Proposition 15, the graph G_0 admits a proper $(6,4)$-coloring.

Assume that for some integer $k \geq 0$, the planar Eulerian triangulation G_k obtained after k applications of the operation $\mathcal{A}$ is $(6,4)$-colorable, and fix such a coloring ϕ_k. Let $G_{k+1} = \mathcal{A}(G_k, f)$ be the triangulation obtained by applying $\mathcal{A}$ to a face f of G_k. We show that G_{k+1} is again $(6,4)$-colorable.

Let the boundary of f be the facial triangle $\{u, v, w\}$, and let the inserted copy of K_3 have vertex set $\{x, y, z\}$. Define $\phi_{k+1} : V(G_{k+1}) \to S$ by $\phi_{k+1}(q) = \phi_k(q)$ for every $q \in V(G_k)$. Since ϕ_k is proper, the colors $\phi_k(u)$, $\phi_k(v)$, and $\phi_k(w)$ are pairwise distinct. Set $T = S \setminus \{\phi_k(u), \phi_k(v), \phi_k(w)\}$, so that $|T| = 3$. Color x, y, and z with distinct elements of T, so that $\phi_{k+1}(x)$, $\phi_{k+1}(y)$, and $\phi_{k+1}(z)$ are pairwise distinct and $T = \{\phi_{k+1}(x), \phi_{k+1}(y), \phi_{k+1}(z)\}$.

For inner vertices x, y, and z, each have exactly four distinct colors in their neighborhoods. For the outer vertices u, v, and w, their neighborhoods have at least four distinct colors due to the adjacent inner vertices having distinct colors assigned.

Therefore, ϕ_{k+1} is a proper $(6,4)$-coloring of G_{k+1}. Hence, by induction on the number of applications of $\mathcal{A}$, every planar Eulerian triangulation obtained from the octahedron by a finite sequence of $\mathcal{A}$-operations is $(6,4)$-colorable. A sample planar Eulerian triangulation colored by ϕ is shown in Figure 15.

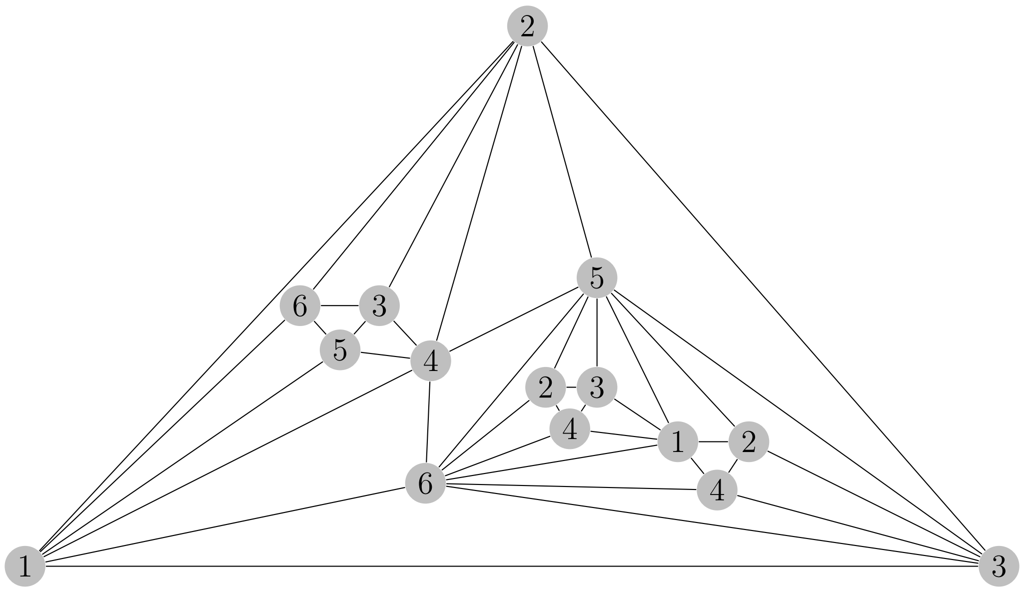


Figure 15. $(6,4)$-coloring ϕ assigned on a planar Eulerian triangulation generated by $\mathcal{A}$-operations.

D. Circulant graphs

Recall we want to prove that, when $p \geq 6$,

$$\chi_2^d(C_p(1,3)) = \begin{cases} 3, & \text{if } p \equiv 0 \pmod 9, \\ 4, & \text{otherwise.} \end{cases}$$

We distinguish cases according to the congruence class of the parameter p.

Case $p \equiv 0 \pmod 9$.

Write each vertex as $i = 3t + r$ with $r \in \{0, 1, 2\}$. Define

$$c(i) = t + r \pmod 3.$$

Let $i = 3t + r$. Then $c(i+3) = t + 1 + r = c(i) + 1 \pmod 3$, and $c(i-3) = t - 1 + r = c(i) + 2 \pmod 3$. Moreover, $c(i+1) = t + (r+1) = c(i) + 1 \pmod 3$, and $c(i-1) = t + (r-1) = c(i) + 2 \pmod 3$. Thus, the neighbors of i receive exactly the two colors different from $c(i)$, and c is a 2-dynamic coloring. Hence, $\chi_2^d(C_p(1,3)) \leq 3$. By 12 $\chi_2^d(C_p(1,3)) \geq 3$ so $\chi_2^d(C_p(1,3)) = 3$. (Figure 16).

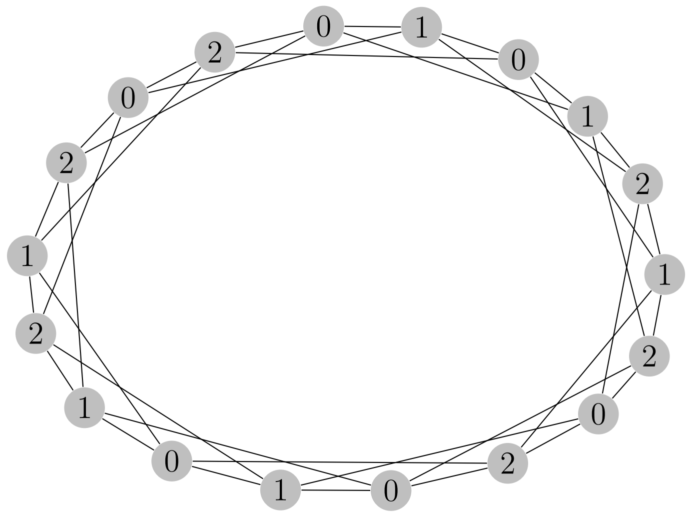


Figure 16. Case $p \equiv 0 \pmod 9$, assignment of coloring c on graph $C_{18}(1,3)$.

Case $p \equiv 0 \pmod 4$.

Define a coloring $\phi : V(C_p(1,3)) \to \{0, 1, 2, 3\}$ by $\phi(i) = i \pmod 4$. If i and j are adjacent, then $j \equiv i \pm 1$ or $j \equiv i \pm 3 \pmod p$. Since $p \equiv 0 \pmod 4$, we have $\phi(j) \equiv i \pm 1$ or $i \pm 3 \pmod 4$, and hence $\phi(j) \neq \phi(i)$. Thus ϕ is proper. For any vertex i, the neighbors $i \pm 1$ and $i \pm 3$ receive the colors $i + 1 \pmod 4$ and $i + 3 \pmod 4$, which are distinct. Thus ϕ is a 2-dynamic coloring and $\chi_2^d(C_p(1,3)) \leq 4$ (Figure 17).

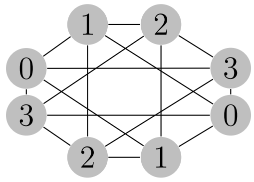


Figure 17. Case $p \equiv 0 \pmod 4$, assignment of coloring ϕ on graph $C_8(1,3)$.

Case $p \equiv 2 \pmod 4$**.**

Define $\phi(i) = i \pmod 4$ for $i \in V(C_p(1,3))$. If i and j are adjacent, then $j \equiv i \pm 1$ or $i \pm 3 \pmod p$. Since $p \equiv 2 \pmod 4$, this implies $j \equiv i \pm 1$ or $i \pm 3 \pmod 4$, and hence $\phi(j) \neq \phi(i)$. Thus ϕ is proper.

Let $i \in V(C_p(1,3))$. The neighbors of i are $i+1$, $i+3$, $i-1$, and $i-3$ modulo p. Their colors are $\phi(i+1) = (i+1) \bmod 4$, $\phi(i+3) = (i+3) \bmod 4$, $\phi(i-1) = (i-1) \bmod 4$, $\phi(i-3) = (i-3) \bmod 4$. Since $(i-1) \bmod 4 = (i+3) \bmod 4 \quad \text{and} \quad (i-3) \bmod 4 = (i+1) \bmod 4$, the neighborhood of i receives exactly the two colors $(i+1) \bmod 4$ and $(i+3) \bmod 4$, which are distinct. Therefore ϕ is a 2-dynamic coloring and $C_p(1,3)$ is $(4,2)$-colorable. An example of this coloring is shown in Figure 18.

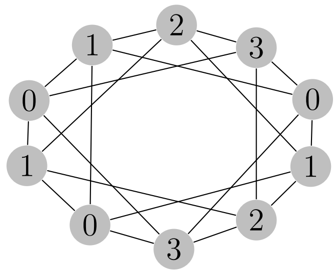


Figure 18. Case $p \equiv 2 \pmod 4 \wedge p \not\equiv 0 \pmod 9$, assignment of coloring ϕ on graph $C_{10}(1,3)$.

Case $p \equiv 1 \pmod 4$ Consider the coloring

$$\phi(i) = \begin{cases} i \bmod 4, & \text{if } i \leq p-5, \\ 3, & \text{if } i = p-4, \\ 2, & \text{if } i = p-3, \\ 3, & \text{if } i = p-2, \\ 1, & \text{if } i = p-1. \end{cases}$$

We prove that ϕ is a proper coloring. Recall that in $C_p(1,3)$ two vertices i and j are adjacent if and only if $j \equiv i \pm 1$ or $i \pm 3 \pmod p$. If $i, j \leq p-5$ and $ij \in E(C_p(1,3))$, then $\phi(i) = i \bmod 4$ and $\phi(j) = j \bmod 4$. Since $j-i \in \{\pm 1, \pm 3\}$ and none of these integers is divisible by 4, it follows that $\phi(i) \neq \phi(j)$. For the remaining edges, at least one endpoint lies in $\{p-4, p-3, p-2, p-1\}$, a direct verification shows that $\phi(i) \neq \phi(j)$ for every such edge. Therefore ϕ is a proper coloring of $C_p(1,3)$. We prove that ϕ is 2-dynamic. Since every vertex of $C_p(1,3)$ has degree 4, it suffices to show that $|\phi(N(i))| \geq 2$ for every vertex i. An example of the color assignment is shown in Figure 19.

Let $0 \leq i \leq p-5$. If $i \leq p-8$, then both $i+1$ and $i+3$ belong to $\{0,\ldots,p-5\}$ and receive colors $(i+1) \bmod 4$ and $(i+3) \bmod 4$, which are distinct since $1 \not\equiv 3 \pmod 4$. Hence $|\phi(N(i))| \geq 2$. If $i \geq 3$, then both $i-1$ and $i-3$ lie in $\{0,\ldots,p-5\}$ and receive distinct colors for the same reason, so again $|\phi(N(i))| \geq 2$. Thus every vertex $i \leq p-5$ has at least two distinct colors in its neighborhood.

For the remaining vertices $i \in \{p-4, p-3, p-2, p-1\}$, a direct inspection shows that their neighbors receive at least two distinct colors. Therefore ϕ is a 2-dynamic coloring of $C_p(1,3)$. This can be observed in Figure 19.

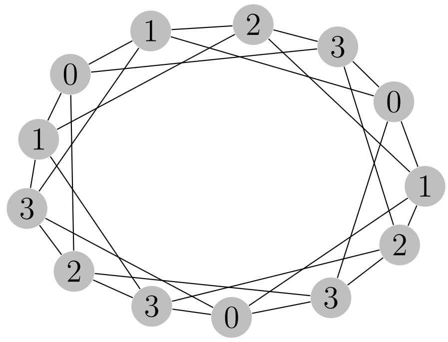


Figure 19. Case $p \equiv 1 \pmod 4 \wedge p \not\equiv 0 \pmod 9$, assignment of coloring ϕ on graph $C_{13}(1,3)$.

Case $p \equiv 3 \pmod 4$. Consider the coloring

$$\phi(i) = \begin{cases} i \bmod 4, & \text{if } 0 \leq i \leq p-4, \\ 2, & \text{if } i = p-3, \\ 0, & \text{if } i = p-2, \\ 1, & \text{if } i = p-1. \end{cases}$$

As in the previous case, $\phi(i) = i \bmod 4$ on $\{0,\ldots,p-4\}$, and therefore every edge with both endpoints in this set is properly colored. The modified colors at the vertices $p-3, p-2, p-1$ eliminate the conflicts arising from edges whose endpoints lie near 0 and p, and a direct verification shows that ϕ is proper. A sample coloring assignment is shown in Figure 20. Similarly, as in the previous case, every vertex $i \leq p-4$ has two neighbors inside $\{0,\ldots,p-4\}$ receiving distinct colors modulo 4, and therefore $|\phi(N(i))| \geq 2$. A direct inspection shows that the same holds for $i \in \{p-3, p-2, p-1\}$. Therefore, ϕ is a 2-dynamic coloring of $C_p(1,3)$. Observe how ϕ assigned on a sample circulant graph in Figure 20 fits $(4,2)$-colorability.

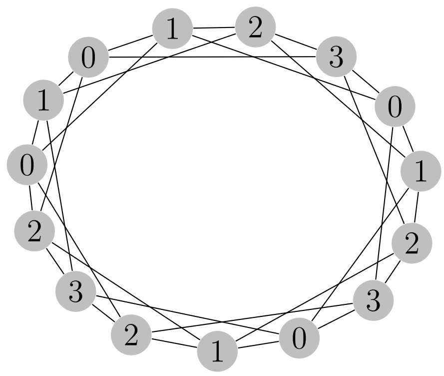


Figure 20. Case $p \equiv 3 \pmod 4 \wedge p \not\equiv 0 \pmod 9$, assignment of coloring ϕ on graph $C_{15}(1,3)$.

We now prove that $\chi_2^d(C_p(1,3)) \geq 4$ whenever $p \not\equiv 0 \pmod 9$. Suppose that $C_p(1,3)$ admits a 2-dynamic 3-coloring ϕ. Let $i \in V(C_p(1,3))$. Since ϕ is proper, $\phi(i) \neq \phi(i \pm 1)$ and $\phi(i) \neq \phi(i \pm 3)$. Moreover, the 2-dynamic condition implies that the colors in $\{\phi(i-1), \phi(i+1), \phi(i-3), \phi(i+3)\}$ are not all equal.

Since only three colors are available and $\phi(i-1) \neq \phi(i+1)$, there is a unique color different from both $\phi(i-1)$ and $\phi(i+1)$. The vertices $i-3$ and $i+3$ must receive this color, and therefore $\phi(i+3) = \phi(i-3)$ for every vertex i. Applying this relation with $i+3$ in place of i gives $\phi(i+6) = \phi(i)$, and applying it once more yields $\phi(i+9) = \phi(i)$ for all i. Since the coloring is defined modulo p, we must have $\phi(p) = \phi(0)$, which implies $9 \mid p$. This contradiction shows that no 2-dynamic 3-coloring exists when $p \not\equiv 0 \pmod 9$, and therefore $\chi_2^d(C_p(1,3)) \geq 4$.

Now, for the proof of Corollary 11, note that every $(1,3)$-circulant graph G has $\chi(G) \geq 2$. By Theorem 10, $\chi_2^d(G) \leq 4$, and thus $\chi_2^d(G) \leq \chi(G) + 2$.

E. Integer linear programming formulations

The integer linear programming model (ASS) mentioned by [Jabrayilov and Mutzel 2018] considers a graph $G = (V, E)$ and an integer k, and solves the proper coloring problem in graphs. It achieves finding a color $c \in C$ for each vertex $v \in V$, where $C = \{0, 1, \ldots, k-1\}$. To achieve this, they define the variables x_{vc} for each vertex v and for each color c, where $x_{vc} = 1$ when v is assigned the color c and $x_{vc} = 0$ otherwise. Additionally, they define the variable w_c for each color c where $w_c = 1$ if c is used in the coloring and $w_c = 0$ otherwise.

E.1. Results

We extend this model to handle r-dynamic colorings by the definition of variables q_{vc} for each vertex v and color c, where $q_{vc} = 1$ when the color assignment to the neighborhood of v includes color c, and $q_{vc} = 0$ otherwise. Therefore, the (ACR) model is defined by

(ACR)

$$\begin{aligned}
\text{minimize} \quad & \sum_{c \in C} w_c, && && (1)\\
\text{subject to} \quad & \sum_{c \in C} x_{vc} = 1 && \forall v \in V, && (2)\\
& x_{uc} + x_{vc} \leq w_c && \forall uv \in E, \forall c \in C, && (3)\\
& w_c \leq \sum_{v \in V} x_{vc} && \forall c \in C, && (4)\\
& w_c \leq w_{c-1} && \forall c \in C - \{0\}, && (5)\\
& \sum_{c \in C} q_{vc} \geq \min\{r, d(v)\} && \forall v \in V, && (6)\\
& \sum_{u \in N(v)} x_{uc} \geq q_{vc} && \forall v \in V, \forall c \in C, && (7)\\
& q_{vc} \geq x_{uc} && \forall v \in V, \forall u \in N(v), \forall c \in C && (8)\\
& x_{vc}, w_c, q_{vc} \in \{0, 1\} && \forall v \in V, \forall c \in C. &&
\end{aligned}$$

The objective function (1) minimizes the number of colors used. Equation (2) restricts that only one color can be assigned to a vertex. For each pair of adjacent vertices, equation (3) prevents both vertices from receiving the same color if it is used in the coloring. When a color is used, equation (4) forces at least one vertex to use it. Equation (5) reduces the complexity of the program by discarding assignments whose structure varies by the colors used. For the neighborhood of each vertex, equation (6) forces a minimum number of colors to be assigned. Equation (7) forces the neighborhood of each vertex to present at least $\min\{r, d(v)\}$ assigned colors. Finally, equation (8) ensures that the neighborhood of a vertex presents a color if at least one adjacent vertex has the same color.

Another approach consists of finding the smallest upper bound for $\chi_r^d(G)$ by taking the solution of the program for a subgraph $G' = (V', E')$ of G and finding the solution for G. Let x' be a binary matrix indexed by V' in the rows and C in the columns. Let w' be a vector of size $|C|$ of binary values. We propose the following (ACR-R) model:

(ACR-R)

$$
\begin{array}{lll}
\text{minimize} & \displaystyle\sum_{c\in C} w_c, & \\
\text{subject to} & \displaystyle\sum_{c\in C} x_{vc} = 1 & \forall v \in V, \\
& x_{vc} = x'_{vc} & \forall v \in V', \forall c \in C, \qquad (9) \\
& x_{uc} + x_{vc} \leq w_c & \forall uv \in E, \forall c \in C, \\
& w_c = w'_c & \forall c \in C \text{ such that } w'_c = 1, \qquad (10) \\
& w_c \leq \displaystyle\sum_{v\in V} x_{vc} & \forall c \in C, \\
& w_c \leq w_{c-1} & \forall c \in C - \{0\}, \\
& q_{vc} = q'_{vc} & \forall v \in V', \forall c \in C \text{ such that } q'_{vc} = 1, \qquad (11) \\
& \displaystyle\sum_{c\in C} q_{vc} \geq \min\{r, d(v)\} & \forall v \in V, \\
& \displaystyle\sum_{u\in N(v)} x_{uc} \geq q_{vc} & \forall v \in V, \forall c \in C, \\
& q_{vc} \geq x_{uc} & \forall v \in V, \forall u \in N(v), \forall c \in C, \\
& x_{vc}, w_c, q_{vc} \in \{0,1\} & \forall v \in V, \forall c \in C.
\end{array}
$$

Regarding equation (9), we assign the value of x'_{vc} found previously using any of the listed models. On the other hand, in equation (10), the value w'_c is only assigned if $w'_c = 1$. This model shows ease in finding an optimal solution because it has a reduced number of vertices.