On Rainbow Coloring of Quadrilateral Snake Graphs[‡]

Felipe B. Wendt[1], Aleffer Rocha[1,2], M. M. Omai[3]

[1]Federal Institute of Paraná (IFPR) – Pinhais, PR – Brazil

[2]Federal University of Paraná (UFPR) – Curitiba, PR – Brazil

[3]Federal Center for Technological Education Celso Suckow da Fonseca (CEFET-RJ) – Rio de Janeiro, RJ – Brazil

20241pin10020014@estudantes.ifpr.edu.br, aleffer.rocha@ifpr.edu.br,

mayara.omai@cefet-rj.br

Abstract. *A path is said rainbow if its edges have pairwise distinct colors. A graph G admits a rainbow coloring if there exists a rainbow path between every pair of vertices of G. When such a path is a shortest path, we say that G has a strong rainbow coloring. The rainbow connection number $rc(G)$ (respectively, the strong rainbow connection number $src(G)$) is the minimum number of colors required to obtain a (strong) rainbow coloring of G. In this work, we determine $rc(G)$ and $src(G)$ when G is a quadrilateral snake graph. We also determine $rc(G)$ when G is a double quadrilateral snake graph.*

1. Introduction

The concept of rainbow coloring was introduced by [Chartrand et al. 2008] and has since attracted attention in several contexts and problems in Graph Theory [Ma et al. 2025, Ma and Zhao 2025, Yulianti et al. 2025], with applications in information security and network routing [Li and Sun 2012]. Let G be a connected simple graph. A *rainbow path* is a path whose edges have pairwise distinct colors. An edge coloring of G is called a *rainbow coloring* if every pair of vertices of G is connected by a rainbow path. The *rainbow connection number* of G, denoted by $rc(G)$, is the minimum number of colors in a rainbow coloring of G.

A stronger version of this coloring requires that at least one rainbow path between every pair of vertices is also a shortest path. In this case, the coloring is called a *strong rainbow coloring*, and the minimum number of colors required for such a coloring is the *strong rainbow connection number* of G, denoted by $src(G)$. The diameter of G, denoted by $diam(G)$, is the maximum distance between any pair of vertices. It is known that $\mathrm{diam}(G) \leq rc(G) \leq src(G) \leq n-1$ [Chartrand et al. 2008].

Determining whether a graph G satisfies $rc(G) \leq k$ for a fixed $k \geq 2$ is an $\mathcal{NP}$-complete problem [Chakraborty et al. 2011, Ananth et al. 2011, Li et al. 2015]. Similarly, for $k \geq 3$, deciding whether $src(G) \leq k$ is also $\mathcal{NP}$-complete [Ananth et al. 2011]. Due to the computational difficulty of these problems, much of the research focuses on determining bounds or exact values of the rainbow connection parameters for specific classes of graphs [Rocha et al. 2020a, Rocha et al. 2022, Yulianti et al. 2025]. It is known that for complete graphs K_n ($n \geq 2$), trees T_n, and cycles C_n ($n \geq 4$), the rainbow connection number coincides with the strong rainbow connection number, given by

[‡]Partially supported by CNPq Proc. 405620/2025-0.

$rc(K_n) = src(K_n) = 1$, $rc(T_n) = src(T_n) = n - 1$, and $rc(C_n) = src(C_n) = \lceil n/2 \rceil$. Some results concerning rainbow connection parameters have been obtained for subclasses of snake graphs. [Rocha et al. 2020a] determined the rainbow connection number of triple triangular snake graphs $S(3, 3, n)$. Later, [Rocha 2020] extended this result to the strong version, showing that $src(S(3, 3, n)) = rc(S(3, 3, n)) = \text{diam}(S(3, 3, n))$ if $n \geq 2$ and $n \neq 3$, and $src(S(3, 3, n)) = rc(S(3, 3, n)) = \text{diam}(S(3, 3, n)) + 1$ if $n = 3$. For general snake graphs, [Rocha et al. 2020b] proved that $rc(S(\ell, k, n)) \leq \text{diam}(S(\ell, k, n)) + 1$ when ℓ is even, and $rc(S(\ell, k, n)) \leq \text{diam}(S(\ell, k, n)) + 2$ when ℓ is odd.

In this work, we determine the rainbow connection number of quadrilateral snake graphs $S(4, 1, n)$ and double quadrilateral snake graphs $S(4, 2, n)$, as well as the strong rainbow connection number of $S(4, 1, n)$, showing that these parameters coincide with the diameter of the graph.

2. Preliminaries

A *subdivision* of an edge $\{u, v\}$ is obtained by deleting $\{u, v\}$ and introducing a new vertex w adjacent to both u and v. Subdividing an edge $\{u, v\}$ of a graph G exactly t times consists of deleting $\{u, v\}$, taking the disjoint union of G with a path $P_t = (w_0, \ldots, w_{t-1})$, and adding the edges $\{u, w_0\}$ and $\{w_{t-1}, v\}$.

Let $\ell \geq 3$, $k \geq 1$, and $n \geq 2$. An *ℓ-lateral k-multiple snake graph* on n vertices, denoted by $S(\ell, k, n)$, is obtained from the path $P_n = (v_0, \ldots, v_{n-1})$, called the *spine*, by adding k parallel edges between each pair v_i and v_{i+1}, $0 \leq i \leq n-2$, and then subdividing each of these added edges exactly $\ell - 2$ times.

A *block* $B_{i,i+1}$ is the subgraph of $S(\ell, k, n)$ induced by v_i, v_{i+1}, and all new vertices from the subdivisions of the edges between v_i and v_{i+1}. There are $k + 1$ internally disjoint paths between v_i and v_{i+1} in the block $B_{i,i+1}$. These paths are denoted by $P_{i,j}$, $0 \leq j \leq k$, so that $P_{i,0}$ is the a path consisting of the single edge $\{v_i, v_{i+1}\}$. The vertices of each path $P_{i,j}$ are labeled as $(u^0_{i,j}, u^1_{i,j}, \ldots, u^{\ell-3}_{i,j})$, for $1 \leq i \leq n - 2$ and $1 \leq j \leq k$.

Since the diameter of a graph provides a natural lower bound for both $rc(G)$ and $src(G)$, we recall the following known result on the diameter of snake graphs.

Lemma 1 ([Rocha et al. 2020b]). *Let $G = S(\ell, k, n)$ be a snake graph, where $\ell \geq 3$, $k \geq 1$, and $n \geq 2$. Then diam$(G) = \lfloor \ell/2 \rfloor$ if $n = 2$ and $k = 1$; $\ell - 1$ if $n = 2$ and $k > 1$; and $2\lfloor \ell/2 \rfloor + n - 3$ if $n > 2$.*

3. Results

In this section we determine the rainbow and strong rainbow connection numbers of quadrilateral snake graphs $S(4, 1, n)$, and the rainbow connection number of double quadrilateral snake graphs $S(4, 2, n)$, for $n \geq 2$.

Theorem 2. *Let G be a quadrilateral snake graph $S(4, 1, n)$ with $n \geq 2$. Then $src(G) = rc(G) = diam(G)$.*

Proof. Let G be a quadrilateral snake graph $S(4, 1, n)$ with $n \geq 2$ be a quadrilateral snake graph. To prove that $rc(G) = diam(G)$, we construct a rainbow coloring of G using exactly $diam(G)$ colors.

For $n = 2$, G is isomorphic to C_4 and $diam(G) = 2$. A strong rainbow coloring with two colors is depicted in Figure 1.

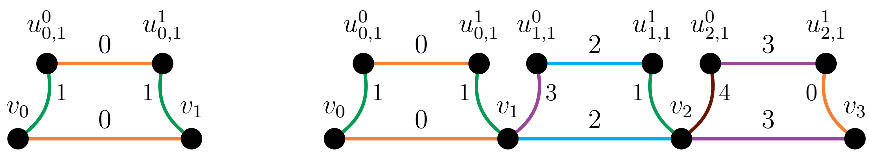

Figure 1. Strong rainbow colorings of $S(4,1,2)$ and $S(4,1,4)$, respectively.

For $n \geq 3$, we have $diam(G) = n+1$. Define a coloring $c : E(G) \to \{0,1,\dots,n\}$ as follows: $c(\{v_0,v_1\}) = 0$; $c(\{v_i,v_{i+1}\}) = i+1$, for $1 \leq i \leq n-2$; $c(\{u^0_{0,1},u^1_{0,1}\}) = 0$; $c(\{u^0_{i,1},u^1_{i,1}\}) = i+1$, for $1 \leq i \leq n-2$; $c(\{v_i,u^0_{i,1}\}) = i+2$, for $1 \leq i \leq n-1$; $c(\{u^1_{n-2},v_{n-1}\}) = 0$; and $c(e) = 1$ for all remaining edges.

Thus, $n+1$ colors are used and, by Lemma 1, $diam(G) = n+1$ for $n \geq 3$. It remains to show that this is a strong rainbow coloring of G. To this end, for every pair of vertices x and y of G, we exhibit a shortest rainbow path P between them.

Case 1. x and y belong to the spine. Any subpath $(v_p, v_{p+1},\dots,v_q)$ of the spine is rainbow, since, by construction, $c(\{v_0,v_1\}) = 0$ and $c(\{v_i,v_{i+1}\}) = i+1$ for $1 \leq i \leq n-2$, so the edges of the spine receive pairwise distinct colors. Thus, a rainbow path between x and y is the subpath $P = (x,\dots,y)$ of the spine. Moreover, this path is shortest, since any path between x and y that leaves the spine is longer.

Case 2. $x,y \in V(B_{i,i+1})$, i.e., x and y belong to the same block. If both belong to the same $P_{i,j}$, then they are adjacent and $P = (x,y)$ is a rainbow path, which is clearly shortest. Otherwise, consider the following cases.

Case 2.1 v_i to $u^1_{i,1}$. Let $P = (v_i, v_{i+1}, u^1_{i,1})$. By construction, the colors used on P are 0 and 1 for $i = 0$, $i+1$ and 1 for $1 \leq i \leq n-3$, and $i+1$ and 0 for $i = n-2$. This path is shortest, since v_i and $u^1_{i,1}$ are not adjacent and therefore there is no shorter path.

Case 2.2 $u^0_{i,1}$ to v_{i+1}. Let $P = (u^0_{i,1}, u^1_{i,1}, v_{i+1})$. By construction, for $1 \leq i \leq n-3$, colors $i+1$ and 1 are used; for $i = 0$, colors 1 and 0 are used; and for $i = n-2$ colors 0 and $n-1$. This path is shortest, since $u^0_{i,1}$ and v_{i+1} are not adjacent and therefore no shorter path exists.

Case 3. $x \in B_{h,h+1}$ and $y \in B_{i,i+1}$ with $h < i$, i.e., x and y belong to distinct blocks.

Case 3.1. Exactly one of x, y belongs to the spine. Without loss of generality, suppose that x belongs to the spine, so $x = v_h$ or $x = v_{h+1}$. Let $P_1 = (x,\dots,v_i)$ be a subpath of the spine, and let P_2 be the shortest path from v_i to y described in Case 2.2. Define P as the concatenation of P_1 and P_2. By Case 1, P_1 is rainbow and uses only colors in $\{0,\dots,i\}$. By Case 2.2, P_2 is rainbow and uses only colors $i+1$ and $i+2$. Hence no color of P_2 appears in P_1, and P is rainbow. Moreover, any path between distinct blocks intersects the spine, so P is shortest.

Case 3.2. Neither x nor y belongs to the spine. Since neither x nor y belongs to the spine, we have $x \in \{u^0_{h,1}, u^1_{h,1}\}$ and $y \in \{u^0_{i,1}, u^1_{i,1}\}$. First consider $x = u^0_{h,1}$ and $y = u^1_{i,1}$. Let $P_1 = (u^0_{h,1}, u^1_{h,1}, v_{h+1})$ (Case 2.2), let $P_2 = (v_{h+1},\dots,v_i)$ be a subpath of the spine (Case 1), and let $P_3 = (v_i, u^0_{i,1}, u^1_{i,1})$ (Case 2.1). Define P as the concatenation $P = P_1 \cdot P_2 \cdot P_3$, where $h < i$. By Case 2.2, P_1 is a rainbow shortest path using colors $\{h+1, 1\}$. By Case 2.1, P_3 is a rainbow shortest path using colors $\{i+1, i+2\}$. By Case 1, P_2 is a shortest path using colors in $\{h+2,\dots,i-1\}$. These sets are pairwise disjoint, hence P is a rainbow shortest path. Finally, if $x = u^1_{h,1}$ we remove the first edge of P_1, and if $y = u^0_{i,1}$ we remove the last edge of P_3. In each case we obtain a subpath of P connecting x and y, which remains a rainbow shortest path.

Therefore, c is a strong rainbow coloring, and hence $src(G) = rc(G) = diam(G)$. $\square$

Theorem 3. *Let G be a double quadrilateral snake graph $S(4, 2, n)$ with $n \geq 2$. Then $rc(G) = diam(G)$.*

Sketch of proof. Let G be a double quadrilateral snake graph $S(4, 2, n)$ with $n \geq 2$. Since $diam(G) \leq rc(G)$, it suffices to construct a rainbow coloring of G using exactly $diam(G)$ colors, as given by Lemma 1. For $n = 2$, $diam(G) = 3$ and a rainbow coloring with $diam(G)$ colors is shown in Figure 2.

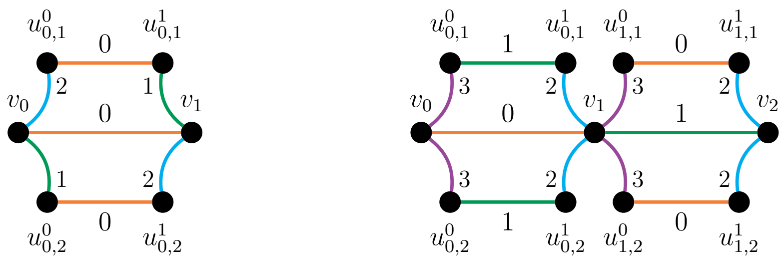

Figure 2. Rainbow colorings of $S(4, 2, 2)$ and $S(4, 2, 3)$, respectively.

For $n \geq 3$, we have $diam(G) = n + 1$. Define a coloring $c : E(G) \rightarrow \{0, 1, \ldots, n\}$ as follows: $c(\{v_i, u^0_{i,j}\}) = n$, for $0 \leq i < n - 1$ and $j \in \{1, 2\}$; $c(\{v_0, v_1\}) = 0$ and $c(\{u^0_{i,j}, u^1_{i,j}\}) = 0$, for $1 \leq i < n - 1$ and $j \in \{1, 2\}$; $c(\{u^0_{0,j}, u^1_{0,j}\}) = 1$, for $j \in \{1, 2\}$, and $c(\{v_{n-2}, v_{n-1}\}) = 1$; $c(\{v_i, v_{i+1}\}) = i + 2$, for $1 \leq i < n - 2$; and $c(\{u^1_{i,j}, v_{i+1}\}) = 2$, for $0 \leq i < n - 1$ and $j \in \{1, 2\}$.

We now show that c is a rainbow coloring by exhibiting a rainbow path P between every pair of vertices. Let x and y be arbitrary vertices of G.

Case 1. x and y belong to the spine. Any subpath $(v_p, v_{p+1}, \ldots, v_q)$ of the spine is rainbow, since the edges of the spine receive pairwise distinct colors: $c(\{v_0, v_1\}) = 0$, $c(\{v_{n-2}, v_{n-1}\}) = 1$, and $c(\{v_i, v_{i+1}\}) = i + 2$ for $1 \leq i \leq n - 2$. Thus, a rainbow path between x and y is the subpath $P = (x, \ldots, y)$ of the spine.

Case 2. $x, y \in V(B_{i,i+1})$. Consider the paths $P' = (u^1_{i,1}, u^0_{i,1}, v_i, v_{i+1}, u^1_{i,2})$ and $P'' = (u^0_{i,1}, u^1_{i,1}, v_{i+1}, v_i, u^0_{i,2})$. By construction, the edges of both P' and P'' receive colors in $\{0, 1, n, 2, i + 2\}$, which are pairwise distinct, so they are rainbow paths. Hence, for any pair of vertices in $B_{i,i+1}$, there exists a rainbow path between them.

Case 3. $x \in V(B_{h,h+1})$ and $y \in V(B_{i,i+1})$ with $h < i$. Any path between distinct blocks must contain a vertex of the spine.Let P be the concatenation of three paths: a path P_1 from x to a spine vertex in $B_{h,h+1}$, a spine subpath P_2, and a path P_3 from a spine vertex to y in $B_{i,i+1}$. By Case 2, P_1 and P_3 use colors from $\{0, 1, 2, n\}$, while by Case 1 P_2 uses colors from $\{0, 1, 3, \ldots, n - 1\}$.

Since $n \geq 3$, the colors 2 and n do not appear in P_2, so no conflict arises with these colors. The colors 0 and 1 appear only on the extremal edges $\{v_0, v_1\}$ and $\{v_{n-2}, v_{n-1}\}$ of the spine, and hence occur in at most one of the subpaths. Therefore, the colors of P_1, P_2, and P_3 are pairwise distinct, and $P = P_1 \cdot P_2 \cdot P_3$ is a rainbow path.

Thus, c is a rainbow coloring of G with $diam(G)$ colors, and $rc(G) = diam(G)$. $\square$

References

Ananth, P., Nasre, M., and Sarpatwar, K. K. (2011). Rainbow connectivity: Hardness and tractability. In Chakraborty, S. and Kumar, A., editors, *IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS 2011)*, volume 13 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 241–251, Dagstuhl, Germany. Schloss Dagstuhl – Leibniz-Zentrum für Informatik.

Chakraborty, S., Fischer, E., Matsliah, A., and Yuster, R. (2011). Hardness and algorithms for rainbow connection. *Journal of Combinatorial Optimization*, 21(3):330–347.

Chartrand, G., Johns, G. L., McKeon, K. A., and Zhang, P. (2008). Rainbow connection in graphs. *Mathematica Bohemica*, 133(1):85–98.

Li, S., Li, X., and Shi, Y. (2015). Note on the complexity of deciding the rainbow (vertex-) connectedness for bipartite graphs. *Applied Mathematics and Computation*, 258:155–161.

Li, X. and Sun, Y. (2012). *Rainbow Connections of Graphs*. Springer US.

Ma, Y., Xue, Y., and Zhang, X. (2025). Proper (strong) rainbow connection and proper (strong) rainbow vertex connection of graphs with large clique number. *Journal of Interconnection Networks*, 25(01):2350034.

Ma, Y. and Zhao, Y. (2025). Some results on (1,2)-rainbow connection number. *Discrete Applied Mathematics*, 373:231–238.

Rocha, A. (2020). Coloração arco-íris em classes de grafos. Master thesis. Federal University of Technology – Paraná (UTFPR), Ponta Grossa.

Rocha, A., Almeida, S., and Zatesko, L. (2020a). The rainbow connection number of triangular snake graphs. In *Anais do V Encontro de Teoria da Computação*, pages 65–68, Porto Alegre, RS, Brasil. SBC.

Rocha, A., Almeida, S. M., and Zatesko, L. M. (2020b). A near-tight bound for the rainbow connection number of snake graphs. In *Proc. IX Latin American Workshop on Cliques in Graphs (LAWCG '20)*, page 12.

Rocha, A., Almeida, S. M., and Zatesko, L. M. (2022). Rainbow connectivity and rainbow criticality on graph classes. *Discrete Applied Mathematics*, 323:311–323. LAGOS'19 - X Latin and American Algorithms, Graphs, and Optimization Symposium - Belo Horizonte, Minas Gerais, Brazil.

Yulianti, L., Nazra, A., et al. (2025). On the rainbow connection numbers of line, middle, and total graphs of wheels. *Electronic Journal of Graph Theory & Applications*, 13(1).