On the (2,1)-total number of indifference graphs

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Abstract. *A k-$(2,1)$-total labelling of a graph G is a function $\pi : V(G) \cup E(G) \to \{0,\ldots,k\}$ such that: $\pi(u) \neq \pi(v)$ for every $uv \in E(G)$; $\pi(uv) \neq \pi(vw)$ for every pair of adjacent edges $uv, vw \in E(G)$; and $|\pi(uv)-\pi(u)| \geq 2$ and $|\pi(uv)-\pi(v)| \geq 2$ for every $uv \in E(G)$. The least integer k for which G admits a k-$(2,1)$-total labelling is denoted by $\lambda_2^t(G)$. In this work, we establish tight upper bounds for $\lambda_2^t(G)$ when G is an indifference graph, according to the parity of $\Delta(G)$. We also determine the $(2,1)$-total number of powers of paths P_n^l with $n = 2l$ vertices, a subclass of indifference graphs.*

1. Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The maximum degree of G is denoted by $\Delta(G)$, or simply Δ if no ambiguity arises, and any vertex of degree Δ is called a *Δ-vertex*. A k-$(2,1)$-*total labelling* of G is a function $\pi : V(G) \cup E(G) \to \{0,\ldots,k\}$ such that $\pi(u) \neq \pi(v)$ for every $uv \in E(G)$; $\pi(uv) \neq \pi(vw)$ for every pair of adjacent edges $uv, vw \in E(G)$; and $|\pi(uv)-\pi(u)| \geq 2$ and $|\pi(uv)-\pi(v)| \geq 2$ for every $uv \in E(G)$. The least integer k for which G admits such a labelling is called $(2,1)$-*total number* of G and is denoted by $\lambda_2^t(G)$.

The $(2,1)$-total labelling was introduced by [Havet and Yu 2008] as a variant of the $L(2,1)$-labelling, a concept originated in the context of channel assignment problems [Hale 1980, Griggs and Yeh 1992]. Havet and Yu established lower and upper bounds for $\lambda_2^t(G)$ of graphs G, proving that $\Delta+1 \leq \lambda_2^t(G) \leq 2\Delta+1$. These bounds were subsequently refined: for graphs with $\Delta \geq 2$, the upper bound improves to $\lambda_2^t(G) \leq 2\Delta$, while for graphs with $\Delta \geq 5$ that are odd, the bound $2\Delta - 1$ holds. Also in their work, Havet and Yu conjectured that $\lambda_2^t(G) \leq 5$ when $\Delta \leq 3$ (except for K_4), and $\lambda_2^t(G) \leq \Delta + 3$ when $\Delta > 3$.

A *clique* is a subset of pairwise adjacent vertices, and a *maximal clique* is one not properly contained in any other. An *indifference graph* is a graph whose vertices can be arranged in a linear order such that the vertices of any maximal clique appear consecutively in the order. This ordering is called an *indifference order*. Also known as *unit interval graphs* and *proper interval graphs*, indifference graphs constitute a natural subclass of interval graphs, a foundational class of intersection graphs with deep structural theory and broad applicability.

In this work, we present tight upper bounds for $\lambda_2^t(G)$ of indifference graphs G, according to the parity of Δ: if Δ is even, then $\lambda_2^t(G) \leq \Delta + 2$, with equality when G has three pairwise adjacent Δ-vertices; otherwise $\lambda_2^t(G) \leq \Delta + 3$, with $\lambda_2^t(G) = \Delta + 2$ when $\Delta \in \{5, 7\}$. These results verify the conjecture of Havet and Yu for indifference graphs. In particular, for power of paths P_n^l with $n \in \{2l+1, 2l+2\}$, [Omai et al. 2022] proved that $\lambda_2^t(P_n^l) = \Delta + 1$; for $n \geq 2l + 3$, these authors also showed that $\lambda_2^t(P_n^l) = \Delta + 2$. In this work, we complement this result by showing that, for $n = 2l$, $\lambda_2^t(P_n^l) = \Delta + 2$ when $l = 2$, and $\lambda_2^t(P_n^l) = \Delta + 1$ when $l \geq 3$.

2. Results

We start this section by stating two results important to our work. In Proposition 1 we provide a lower bound for the $(2, 1)$-total number of graphs having three pairwise adjacent Δ-vertices whose proof was omitted due to space limitations. Theorem 2 [Chia et al. 2013, Havet and Yu 2008] establishes $\lambda_2^t(K_n)$.

Proposition 1. *Let G be a simple graph. If G has at least three pairwise adjacent Δ-vertices, then $\lambda_2^t(G) \geq \Delta + 2$.* □

Theorem 2. *If K_n is a complete graph with $n \geq 2$, then: $\lambda_2^t(K_n) = \Delta + 3$ if $n = 4$ or $n \geq 10$ is even; and $\lambda_2^t(K_n) = \Delta + 2$ otherwise.* □

Now, we present upper bounds for the $(2, 1)$-total number of indifference graphs, depending on the parity of Δ, identifying the cases in which these bounds are tight.

Theorem 3. *Let G be an indifference graph with maximum degree Δ. Then:*

(i) If Δ is even or $\Delta \in \{5, 7\}$, then $\lambda_2^t(G) \leq \Delta + 2$; moreover, if G has at least three pairwise adjacent Δ-vertices, then $\lambda_2^t(G) = \Delta + 2$;

(ii) If Δ is odd and $\Delta \notin \{5, 7\}$, then $\lambda_2^t(G) \leq \Delta + 3$.

Proof. Let G be an indifference graph with indifference order $v_0, v_1, \ldots, v_{n-1}$. Consider the complete graph $K_{\Delta(G)+1}$ with vertex set $V(K_{\Delta(G)+1}) = \{u_0, u_1, \ldots, u_{\Delta(G)}\}$, and let τ be a k-$(2, 1)$-total labelling of $K_{\Delta(G)+1}$, with k determined by Theorem 2, according to the parity of $\Delta(G) + 1$. Define $f\colon V(G) \to V(K_{\Delta(G)+1})$ as $f(v_i) = u_{i \bmod (\Delta(G)+1)}$ for $0 \leq i \leq n - 1$.

We claim that for every edge $v_iv_j \in E(G)$, we have $f(v_i) \neq f(v_j)$. Indeed, if $f(v_i) = f(v_j)$, then $j \equiv i \pmod{\Delta(G) + 1}$, so $|i - j| \geq \Delta(G) + 1$. But in an indifference graph, each vertex is adjacent only to vertices within distance at most $\Delta(G)$ in the indifference order, so v_i and v_j would be nonadjacent, a contradiction. Therefore, the following function is well-defined: $\pi\colon V(G) \cup E(G) \to \{0, 1, \ldots, k\}$ given by $\pi(v_i) = \tau(f(v_i))$ and $\pi(v_iv_j) = \tau(f(v_i)f(v_j))$.

We now verify that π is a valid $(2, 1)$-total labelling of G. For every edge $v_iv_j \in E(G)$, the inequalities $|\pi(v_i) - \pi(v_iv_j)| \geq 2$ and $|\pi(v_j) - \pi(v_iv_j)| \geq 2$ hold since these labels are inherited from τ. Similarly, no two adjacent edges receive the same label, as this property is also inherited from τ without any relabelling. Finally, adjacent vertices receive distinct labels: if $\pi(v_i) = \pi(v_j)$ with $i \neq j$, then $f(v_i) = f(v_j)$, and by the claim above, v_i and v_j are nonadjacent. Hence π is a valid $(2, 1)$-total labelling of G.

Since π is constructed from τ without any relabelling, the value of k equals $\lambda_2^t(K_{\Delta(G)+1})$. We now determine this value according to the parity of $\Delta(G) + 1$.

Case 1: $\boldsymbol{\Delta(G)}$ even. By Theorem 2, $\lambda_2^t(K_{\Delta(G)+1}) = \Delta(G) + 2$. Hence π is a $(\Delta + 2)$-$(2, 1)$-total labelling, proving $\lambda_2^t(G) \leq \Delta + 2$.

Case 2: $\boldsymbol{\Delta(G)}$ odd. By Theorem 2, for $\Delta(G) \in \{5, 7\}$, we have $\Delta(G) + 1 \in \{6, 8\}$ and, thus, $\lambda_2^t(K_{\Delta(G)+1}) = \Delta(G) + 2$. Therefore, π is a $(\Delta + 2)$-$(2, 1)$-total labelling, proving $\lambda_2^t(G) \leq \Delta + 2$. If $\Delta \notin \{5, 7\}$, then $\lambda_2^t(K_{\Delta(G)+1}) = \Delta(G) + 3$, so π is a $(\Delta + 3)$-$(2, 1)$-total labelling, proving $\lambda_2^t(G) \leq \Delta + 3$.

Finally, in Case 1, and in Case 2 when $\Delta \in \{5, 7\}$, if G contains at least three pairwise adjacent Δ-vertices, then by Proposition 1, $\lambda_2^t(G) = \Delta + 2$. □

For a simple graph G, we denote $dist_G(u, v)$ the *distance* between u and v in G. The *l-th power of paths* P_n^l is the simple graph with vertex set $V(P_n)$ and two distinct vertices u and v being adjacent if, and only if, $dist_{P_n}(u, v) \leq l$. Note that when $l = 1$, graph $P_n^l \cong P_n$ and when $n \leq l + 1$, $P_n^l \cong K_n$. Moreover, since powers of paths are indifference graphs, Theorem 3 provides upper bounds for $\lambda_2^t(P_n^l)$ and, in particular, generalizes a result from [Omai et al. 2022], as for $n \geq 2l + 3$, P_n^l has three pairwise adjacent Δ-vertices and Δ even. These results are summarized in Corollary 4.

Corollary 4. *Let P_n^l be the l-th power of paths. Then,*

$$\lambda_2^t(P_n^l) = \begin{cases} \Delta(P_n^l) + 1, & \text{if } n \in \{2l+1,\, 2l+2\}; \\ \Delta(P_n^l) + 3, & \text{if } n \leq l+1 \text{ and } \big(n = 4 \text{ or } (n \geq 10 \text{ even})\big); \\ \Delta(P_n^l) + 2, & \text{if } n \geq 2l+3 \text{ or } \big(l+1 < n < 2l \text{ with } (n \text{ odd or } n \in \{6, 8\})\big) \\ & \quad \text{or } \big(n \leq l+1 \text{ with } n \neq 4 \text{ and } (n < 10 \text{ or } n \text{ odd})\big). \end{cases}$$

Moreover, if $l + 1 < n \leq 2l$ is even and $n \notin \{6, 8\}$, then $\lambda_2^t(P_n^l) \leq \Delta(P_n^l) + 3$.

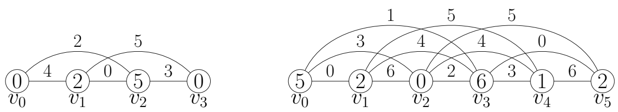
Proof. By Theorems 2-3 and [Omai et al. 2022]. □

The only unsettled instances occur when $l + 1 < n \leq 2l$, n is even, and $n \notin \{6, 8\}$; in these cases, general results for indifference graphs yield just upper bounds. In Theorem 5, for $n = 2l$, we prove that $\lambda_2^t(P_n^l) = \Delta + 2$ when $l = 2$; and $\lambda_2^t(P_n^l) = \Delta + 1$ when $l \geq 3$. In order to prove this result, we fix τ as the standard $(n + 1)$-$(2, 1)$-total labelling of K_n with n odd and $V(K_n) = \{u_0, \ldots, u_{n-1}\}$ as defined in [Chia et al. 2013].

For $n = 3$, $\tau(u_0) = 4$, $\tau(u_1) = 2$, $\tau(u_2) = 0$, $\tau(u_0u_1) = 0$, $\tau(u_1u_2) = 4$, and $\tau(u_2u_0) = 2$. For $n \geq 5$, label the vertices of K_n as $\tau(u_0) = 0$ and $\tau(u_i) = i + 2$ for $1 \leq i \leq n - 1$. For the edges of K_n, define: $E_i' = \{u_{(i+j+1) \bmod n} u_{(i-j-1) \bmod n} : j \in [1, \frac{n-3}{2}]\}$; $E_0 = (E_0' \cup \{u_0u_{n-2}\}) \setminus \{u_2u_{n-2}\}$; $E_1 = E_1' \cup \{u_0u_2\}$; $E_i = E_i'$ for $2 \leq i \leq n - 1$; $E_n = \{u_{(4j-2) \bmod n} u_{(4j) \bmod n} : j \in [1, \frac{n-1}{2}]\}$; and $E_{n+1} = (\{u_{(4j) \bmod n} u_{(4j+2) \bmod n} : j \in [1, \frac{n-1}{2}]\} \cup \{u_2u_{n-2}\}) \setminus \{u_0u_{n-2}\}$. Lastly, $\tau(e) = 3$ if $e \in E_0$; $\tau(e) = 2$ if $e \in E_1$; $\tau(e) = i + 2$ if $e \in E_i$ $(2 \leq i \leq n - 1)$; $\tau(e) = 0$ if $e \in E_n$; and $\tau(e) = 1$ if $e \in E_{n+1}$.

Theorem 5. *Let P_{2l}^l be a power of paths. If $l = 2$, then $\lambda_2^t(P_{2l}^l) = \Delta + 2$; otherwise, $\lambda_2^t(P_{2l}^l) = \Delta + 1$.*

Proof. Consider P_n^l with $n = 2l$ and $l \geq 2$. Let $V(P_n^l) = \{v_0, v_1, \ldots, v_{n-1}\}$. Since $\Delta = n-1$ and $\lambda_2^t(P_n^l) \geq \Delta + 1$, it suffices to construct a n-$(2,1)$-total labelling π for P_n^l to show that $\lambda_2^t(P_n^l) = \Delta + 1$. For $l = 2$, inspection gives $\lambda_2^t(P_4^2) > \Delta + 1$ while Figure 1 (left) establishes $\lambda_2^t(P_4^2) = \Delta + 2$; for $l = 3$, the result follows from the labelling in Figure 1 (right).



(a) (b)

Figure 1. A 5-$(2,1)$-total labelling of P_4^2 (left); a 6-$(2,1)$-total labelling of P_6^3 (right).

Recalling that $n = \Delta + 1$ and $\Delta = 2l - 1$, for $l \geq 4$, consider τ the standard n-$(2,1)$-total labelling of K_{n-1}. Define $f : V(P_n^l) \to V(K_{n-1})$ as follows: $f(v_i) = u_{l-i-1}$ for $0 \leq i \leq l-1$; and $f(v_i) = u_{3l-2-i}$ for $l \leq i \leq n-1$. Now, let $\pi : V(P_n^l) \cup E(P_n^l) \to \{0, 1, \ldots, n\}$ be as follows: for each vertex v_i, $0 \leq i \leq n-1$, let $\pi(v_i) = \tau(f(v_i))$; and for each edge v_iv_j, $0 \leq i < j \leq n-1$ and $|i - j| \leq l$, let $\pi(v_iv_j) = \tau(f(v_i)f(v_j))$. Although π inherits the properties of a $(2,1)$-total labelling from τ, conflicts arise from v_0 and v_{n-1} being mapped to the same vertex of K_{n-1}, namely between the two pairs of edges: **(i)** v_0v_l and v_lv_{n-1}; and **(ii)** v_0v_{l-1} and $v_{l-1}v_{n-1}$.

For conflict **(i)**, since $f(v_0) = f(v_{n-1}) = u_{l-1}$ and $f(v_l) = u_{2l-2}$, both edges v_0v_l and v_lv_{n-1} are mapped to the same edge of K_{n-1}, forcing $\pi(v_0v_l) = \pi(v_lv_{n-1})$. Setting $\pi(v_0v_l) = 3$ resolves this conflict; it remains to verify that no new conflicts arise. First, recall that $\pi(v_0) = \tau(u_{l-1}) = l + 1$ and $\pi(v_l) = \tau(u_{2l-2}) = 2l$. Hence, $|\pi(v_0) - \pi(v_0v_l)| = |l + 1 - 3| = |l - 2| \geq 2$ and $|\pi(v_l) - \pi(v_0v_l)| = |2l - 3| \geq 2$ hold for $l \geq 4$. Next, we verify that v_0v_l is the only edge with label 3 incident with v_0 and v_l in P_n^l. By definition of τ, edges assigned label 3 belong to E_0; in particular, $E_0' = \{u_{(j+1) \bmod (2l-1)}u_{(-j-1) \bmod (2l-1)} : 1 \leq j \leq l-2\}$. Initially, note that there is no $j \in [1, l-2]$ such that $l - 1 = (-j-1) \bmod (2l-1)$; and, for $j = l-2$, we have $l - 1 = (j+1) \bmod (2l-1)$. However, in this last case the edge of K_{2l-1} is $u_{l-1}u_l$, which would correspond to v_0v_{n-2} and $v_{n-1}v_{n-2}$. Since v_0v_{n-2} is the only edge incident with v_0 and $v_0v_{n-2} \notin E(P_n^l)$, the result follows for v_0. Consider now v_l, correspondent to u_{2l-2}. By similar arguments, we conclude that an edge with label 3 incident with u_{2l-2} would only be possible with $j = 0$ or $j = 2l-3$. However, in this case, $0 < 1 \leq j \leq l - 2 < 2l - 3$ for $l \geq 4$, showing no new conflicts arise.

For solving conflict **(ii)**, we assign $\pi(v_{l-1}v_{n-1}) = n = 2l$. Recall that $\pi(v_{l-1}) = \tau(u_0) = 0$ and $\pi(v_{n-1}) = \tau(u_{l-1}) = l + 1$. Hence, $|\pi(v_{l-1}) - \pi(v_{l-1}v_{n-1})| = |0 - 2l| = 2l \geq 2$ and $|\pi(v_{n-1}) - \pi(v_{l-1}v_{n-1})| = |l + 1 - 2l| = l - 1 \geq 2$ hold for $l \geq 4$. Now, we show that, in π, no edge labelled $2l$ is incident with either v_{l-1} or v_{n-1}. By definition of τ edges labelled $2l$ belong to $E_{2l-2} = \{u_{j \bmod (2l-1)}u_{(-j-2) \bmod (2l-1)} : 1 \leq j \leq l-2\}$. By similar arguments used in **(i)**, we conclude that the only edges of E_{2l-2} potentially incident with u_{l-1} occurs when $j = l-2$, yielding edge $u_{l-2}u_{l-1}$, which corresponds to v_1v_0 and v_1v_{n-1}. Since $v_1v_{n-1} \notin E(P_n^l)$, the result follows for v_{n-1}. On the other hand, no edge of E_{2l-2} is incident with u_0 for any $j \in [1, l-2]$ with $j \geq 4$. Therefore, the result follows. $\square$

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