On the conformability of the disjoint union of graphs

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Abstract. *[Chetwynd and Hilton 1988] defined conformable vertex coloring, intending to characterize the vertex coloring induced by a $(\Delta+1)$-total coloring. [Sánchez-Arroyo 1989] proved that determining the total chromatic number is an NP-complete problem even for cubic bipartite graphs. In contrast, [Faria et al. 2025] proved that characterizing conformable graphs with maximum degree three is a polynomial-time problem. The definition of conformable vertex coloring introduces a dependency on the number of connected components. Therefore, the study of conformability of disjoint unions of graphs becomes relevant. In this paper, we determine how conformability behaves under disjoint union, depending on whether the components are conformable or anticonformable.*

1. Introduction

A *k-vertex coloring* of G is an assignment of k colors to the vertices of G so that adjacent vertices have different colors. The colors are denoted by natural numbers and a class of color $i \in \mathbb{N}$ by $\mathcal{C}_i$. The *parity* of $\mathcal{C}_i$ is even (resp. odd), respectively, if $|\mathcal{C}_i|$ is even (resp. odd). The class of color $\mathcal{C}_i$ is an *even color class* (resp. *odd color class*), when the parity of $\mathcal{C}_i$ is even (resp. odd).

A *k-total coloring* of G is an assignment of k colors to the vertices and edges of G so that adjacent or incident elements have different colors. The *total chromatic number* of G, denoted by $\chi''(G)$, is the smallest k for which G has a k-total coloring. Clearly, $\chi''(G) \geq \Delta + 1$. [Behzad 1965] and [Vizing 1964] independently conjectured that the total chromatic number of any graph is at most $\Delta + 2$. The TCC was proved for subcubic graphs [Rosenfeld 1971, Vijayaditya 1971]. If this conjecture holds then the graphs can be partitioned into 2 sets: graphs G such that $\chi''(G) = \Delta(G) + 1$ called *Type 1*, and graphs G such that $\chi''(G) = \Delta(G) + 2$ called *Type 2*.

The *deficiency* of G is $def(G) = \sum_{v \in V} (\Delta(G) - d(v))$, where $d(v)$ is the degree of a vertex v in G. A graph G is *conformable* [Chetwynd and Hilton 1988] if G has a $(\Delta + 1)$-vertex coloring φ in which the number of color classes (including empty color classes) with a parity different from that of $|V(G)|$ is at most $def(G)$. In this case, we say that φ is a *conformable coloring*. If G is not conformable, then G is said *non-conformable*.

Chetwynd and Hilton defined conformable vertex coloring in 1988 [Chetwynd and Hilton 1988], with the aim of characterizing the vertex coloring induced by a $(\Delta + 1)$-total coloring.

Theorem 1.1 (Chetwynd and Hilton [Chetwynd and Hilton 1988])**.** *If G is Type 1, then G is conformable.*

[Hamilton et al. 1999] presented necessary conditions for a graph to be non-conformable. [Hilton and Hind 2002] proved that non-conformable graphs have $def(G) \leq \Delta - 2$. [Campos and de Mello 2007] proposed a conjecture giving a characterization of *Type 1* powers of cycle, stating that all *Type 2* powers of cycle are non-conformable. [Vignesh et al. 2018] conjectured that all line graphs of complete graphs $L(K_n)$ are *Type 1*. [Nigro et al. 2021] proved that there is a circulant graph that is *Type 2* and conformable. [Zorzi et al. 2022] proved that the characterization of [Campos and de Mello 2007] holds for the conformable powers of cycle. Furthermore, they show that it is possible to determine whether a power of cycle is *Type 1* in FPT-time when k (the power of the cycle) is the parameter. [Faria et al. 2023b] proved that the line-graph of K_n is conformable for any $n \geq 1$. Furthermore, they proved that if G is *Class 1*, then $L(G)$ is conformable. In the same year, [Faria et al. 2023a] gave sufficient conditions for an equitable coloring to be conformable. [Faria et al. 2025] characterized that conformable graphs of maximum degree 3 can be recognized in polynomial-time (Theorem 1.2).

Theorem 1.2 ([Faria et al. 2025])**.** *Let G be a graph with maximum degree $\Delta(G) = 3$. Then G is non-conformable if and only if*

1. *G is the disjoint union of an odd number of K_4 or;*
2. *G is the disjoint union of an even number of K_4 with one $K_{3,3}$ or;*
3. *G is the disjoint union of an odd number of K_4 with one triangular prism graph L_3 (see Figure 1c).*

An auxiliary tool, called *anticonformable coloring*, is used to determine the conformability of the disjoint union of graphs and was introduced in [Faria et al. 2025]. A graph G is *anticonformable* if it has a $(\Delta + 1)$-vertex coloring φ such that the number of color classes (including empty color classes) having the same parity as $|V(G)|$ is at most $def(G)$. In this case, we say that φ is an anticonformable coloring. Note that if G is a regular graph, then φ is called anticonformable if each color class has not the same parity than $|V(G)|$. See Figure 1 for examples.

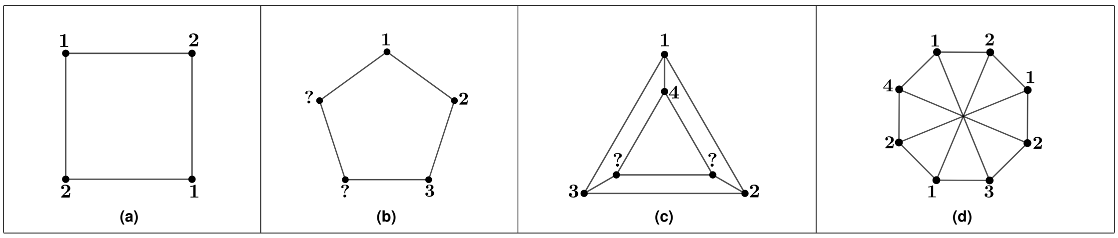

Figure 1. In (1a), the cycle graph C_4 which is conformable, but is *Type 2*. In (1b), the cycle graph C_5 which is non-conformable and, consequently, is *Type 2*. Note that the maximum independent set of C_5 has size 2, which implies that any conformable coloring has each color class $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ with size 1. However, the number of vertices of C_5 is 5. In (1c), the triangular prism graph L_3 is not anticonformable. Note that the maximum independent set of this graph is 2, which implies that any anticonformable coloring has the four color classes with size 1. However, the number of vertices of the graph is 6. In (1d), the Möbius ladder M_8 is anticonformable.

In this paper, we prove that the disjoint union of two conformable graphs, respectively of two anticonformable graphs, results in a conformable graph (Theorems 2.1 and 2.2) and we prove that the disjoint union of a conformable graph and an anticonformable graph results in an anticonformable graph (Theorem 2.3).

2. Main result

In this section, we present results on the disjoint union of conformable and anticonformable graphs, namely Theorems 2.1, 2.3 and 2.2. A straightforward consequence of these theorems is given in Corollary 2.1. For this, we introduce the next definitions and some auxiliary properties (Lemmas 2.1,2.2 and 2.3).

Definition 2.1. *A $(\Delta(G)+1)$-vertex coloring c_G of G with colors $1, 2, \ldots, \Delta(G)+1$ is said to be (k,p)-ordered if it has exactly k colors of parity p; and either $k = 0$ or the colors of c_G that are of parity p are the colors from 1 to k.*

Let $G = (V, E)$ and $G' = (V', E')$ be two graphs, let $0 \leq k \leq def(G)$ and $0 \leq k' \leq defG')$ be two integers, let c_G be a (k,p)-ordered vertex coloring of G and $c_{G'}$ be a (k',p')-ordered vertex coloring of G', where p and p' belong to the set $\{odd, even\}$. Let $\kappa = \min(k,k')$, $K = \max(k,k')$, $\delta = \min(\delta(G), \delta(G'))$, $\Delta = \max(\Delta(G), \Delta(G'))$.

We notice that

$$def(G \cup G') \geq def(G) + def(G') + \Delta - \delta.$$

Let $c_{G\cup G'}$ be the coloring of the disjoint union of G and G' obtained by using c_G to color the vertices of G and $c_{G'}$ to color the vertices of G'.

If $K \leq \delta + 1$, we divide the $\Delta + 1$ colors of $c_{G\cup G'}$ into four sets:

- $E_1 = \{1, \ldots, \kappa\}$ if $\kappa \neq 0$; otherwise, $E_1 = \emptyset$;
- $E_2 = \{\kappa+1, \ldots, K\}$ if $K \neq \kappa$; otherwise, $E_2 = \emptyset$;
- $E_3 = \{K+1, \ldots, \delta+1\}$ if $\delta + 1 \neq K$; otherwise, $E_3 = \emptyset$;
- $E_4 = \{\delta+2, \ldots, \Delta+1\}$ if $\Delta + 1 \neq \delta + 1$; otherwise, $E_4 = \emptyset$.

If $K > \delta + 1$, then we divide the set of the $\Delta + 1$ colors of $c_{G\cup G'}$ into four sets:

- $F_1 = \{1, \ldots, \kappa\}$ if $\kappa \neq 0$; otherwise, $F_1 = \emptyset$;
- $F_2 = \{\kappa+1, \ldots, \delta+1\}$ if $\delta + 1 \neq \kappa$; otherwise, $F_2 = \emptyset$;
- $F_3 = \{\delta+2, \ldots, K\}$ if $K \neq \delta + 1$; otherwise, $F_3 = \emptyset$;
- $F_4 = \{K+1, \ldots, \Delta+1\}$ if $\Delta + 1 \neq K$; otherwise, $F_4 = \emptyset$.

Lemma 2.1.

$$|E_2 \cup E_4| \leq |K - \kappa| + \Delta - \delta \leq \max(def(G), def(G')) + (\Delta - \delta) \leq def(G \cup G')$$

and

$$|F_2 \cup F_3 \cup F_4| \leq K + \Delta - \delta \leq \max(def(G), def(G'))) + (\Delta - \delta) \leq def(G \cup G')$$

Lemma 2.2. *If $K \leq \delta + 1$ then*

1. *if $p = p'$ then $c_{G \cup G'}$ is such that : colors in E_1 and colors in E_3 are even, colors in E_2 are odd. Hence the odd color classes of $c_{G \cup G'}$ are included in the set $E_2 \cup E_4$ which, by Lemma 2.1, has cardinality at most $def(G \cup G')$.*
2. *if $p \neq p'$ then $c_{G \cup G'}$ is such that : colors in E_1 and colors in E_3 are odd, colors in E_2 are even. Hence the even color classes of $c_{G \cup G'}$ are included in the set $E_2 \cup E_4$ which, by Lemma 2.1, has cardinality at most $def(G \cup G')$.*

Lemma 2.3. *If $K > \delta + 1$ then*

1. *if $p = p'$ then $c_{G \cup G'}$ is such that : colors in F_1 are even, colors in F_2 are odd. Hence the odd color classes in $c_{G \cup G'}$ are included in the set $F_2 \cup F_3 \cup F_4$ which, by Lemma 2.1, has cardinality at most $def(G \cup G')$.*
2. *if $p \neq p'$ then $c_{G \cup G'}$ is such that : colors in F_1 are odd, colors in F_2 are even. Hence the even color classes are included in the set $F_2 \cup F_3 \cup F_4$ which, by Lemma 2.1, has cardinality at most $def(G \cup G')$.*

Theorem 2.1. *If G and G' are conformable then $G \cup G'$ is conformable.*

Theorem 2.2. *If G and G' are anticonformable then $G \cup G'$ is conformable.*

Theorem 2.3. *If G is anticonformable and G' is conformable then $G \cup G'$ is anticonformable.*

Sketch of the proof of Theorem 2.1. We consider two cases.

1. The sets of vertices of G and G' have the same parity p. Let p' be the parity different from p.
   Since G and G' are both conformable, there exist integers k and k' such that $k \leq \min(\Delta(G)+1, def(G))$ and $k' \leq \min(\Delta(G')+1, def(G'))$, a (k, p')-ordered $(\Delta(G)+1)$-vertex coloring c_G of G and a (k', p')-ordered $(\Delta(G')+1)$-vertex coloring c'_G of G'. The graph $G \cup G'$ has an even number of vertices. By Lemma 2.2 and Lemma 2.3 the coloring $c_{G \cup G'}$ of $G \cup G'$ obtained by using c_G to color the vertices of G and $c_{G'}$ to color the vertices of G' is such that the number of odd color classes is at most $def(G \cup G')$. Hence $G \cup G'$ is conformable.
2. The sets of vertices of G and G' have distinct parities respectively p and p'.
   Since G and G' are both conformable, there exist integers k and k' such that $k \leq \min(\Delta(G) + 1, def(G))$ and $k' \leq \min(\Delta(G') + 1, def(G'))$, a (k, p')-ordered $(\Delta(G) + 1)$-vertex coloring c_G of G and a (k', p)-ordered $(\Delta(G') + 1)$-vertex coloring c'_G of G'. The graph $G \cup G'$ has an odd number of vertices. By Lemma 2.2 and Lemma 2.3 the coloring $c_{G \cup G'}$ of $G \cup G'$ obtained by using c_G to color the vertices of G and $c_{G'}$ to color the vertices of G' is such that the number of even color classes is at most $def(G \cup G')$. Hence $G \cup G'$ is conformable.

□

Notice that, by exchanging "even" and "odd" as well as $p = p'$ and $p \neq p'$, Case 1 and Case 2 of the proof of Theorem 2.1 are similar ; this comes from the fact that the proofs of Lemma 2.2 and Lemma 2.3 also have this property. The proofs of Theorem 2.2 and Theorem 2.3 can be done using the same sketch.

Corollary 2.1. *Let λ be a positive integer and $\mathcal{G} := \bigcup_{i=1}^{\lambda} G_i$ be an disjoint union of λ of graphs G_i for $i \in \{1, \ldots, \lambda\}$. If $\mathcal{G}$ is neither conformable nor anticonformable, then at least one of the G_i's is non-conformable or non-anticonformable.*

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