Orthogonality between acyclic subdigraphs and paths in digraphs

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Abstract. *Let D be a digraph. A collection of disjoint sets of vertices (respec., collection of disjoint subdigraphs) $\mathcal{H}$ of D and a vertex subset (or subdigraph) Q of D are orthogonal if every set (respec., subdigraph) $H \in \mathcal{H}$ contains exactly one vertex of Q. A well-known result of Gallai and Milgram shows that for every minimum path partition of a digraph there is a stable set orthogonal to it. Similarly, Gallai, Hasse, Roy and Vitaver independently proved that for every longest path of a digraph there is a vertex partition into stable sets (i.e, vertex-coloring) orthogonal to it. Berge showed that no analogous statements hold when optimality is required for the stable set or the vertex coloring. In this paper, we show that this holds if we replace stable sets by induced acyclic subdigraphs.*

1. Introduction

We assume that the reader is familiar with common terminology of graphs and digraphs [Bondy and Murty 2008, Bang Jensen and Gutin 2008]. Let $D = (V, A)$ denote a digraph. The *underlying graph* of D, denoted by $U(D)$, is the simple graph with vertex set $V(D)$ such that u and v are adjacent in $U(D)$ if and only if $(u, v) \in A(D)$ or $(v, u) \in A(D)$. A *stable set* of D is a stable set of $U(D)$. We denote the size of a maximum stable set of D by $\alpha(D)$. A *path partition* of D is a collection of disjoint paths that covers $V(D)$. The size of a minimum path partition of D is denoted by $\pi(D)$. We denote by $\text{ini}(P)$ the initial vertex of a path P. Similarly, if $\mathcal{P}$ is a collection of paths, we denote by $\text{ini}(\mathcal{P})$ the set of initial vertices of the paths in $\mathcal{P}$. A collection of disjoint sets of vertices (respec., collection of disjoint subdigraphs) $\mathcal{H}$ of D and a vertex subset (or subdigraph) Q of D are orthogonal if every set (respec., subdigraph) $H \in \mathcal{H}$ contains exactly one vertex of Q. We also say that $\mathcal{H}$ is *orthogonal to* Q and vice versa.

[Gallai and Milgram 1960] showed that for every minimum path partition of D there is a stable set orthogonal to it.

Theorem 1 (Gallai-Milgram Theorem)**.** *Let D be a digraph. For every minimum path partition $\mathcal{P}$ of D, there exists a stable set S of D such that $\mathcal{P}$ and S are orthogonal. In particular, $\pi(D) \leq \alpha(D)$.*

*Supported by FAPESP (grant 2022/03735-0).
†Supported by FAPESP (grant 2023/03167-5).
‡Supported by FAPESP (grants 2023/03167-5 and 2015/11937-9) and CNPq (grant 302596/2022-4).

One interesting question is if we require optimality on the stable set rather than the path partition, can we obtain a result similar to Theorem 1?

Question 1. *Let D be a digraph and let T be a maximum stable set of D. Is it true that D contains a path partition orthogonal to T?*

[Berge 1982a] observed that the answer to this question is negative by exhibiting orientations of odd cycles that do not satisfy the desired property. See Figure 1a. Note that the set $\{v_3, v_5\}$ is a maximum stable set of the digraph D depicted, but there is no orthogonal path partition to it.

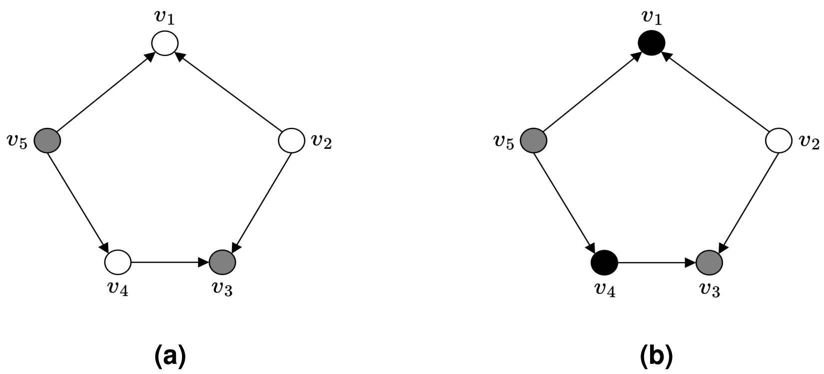

Figure 1. Orientation D of a C_5 that is a counterexample to both Question 1 and 2.

A *coloring* of a digraph D is a coloring of $U(D)$, that is, a partition of $V(D)$ into stable sets. The *chromatic number* of D, denoted by $\chi(D)$, is the chromatic number of $U(D)$. We denote by $\lambda(D)$ the order of a longest path of D. We may obtain *dual* results by exchanging the roles of stable sets and paths. In other words, we replace path partition by (vertex)-coloring and replace stable set by path.

In the 1960s, a result analogous to the Gallai-Milgram Theorem was proved independently by [Gallai 1968], [Hasse 1965], [Roy 1967] and [Vitaver 1962].

Theorem 2 (Gallai-Hasse-Roy-Vitaver Theorem)**.** *] Let D be a digraph. For every maximum path P of D, there exists a coloring S of D such that P and S are orthogonal. In particular, $\chi(D) \leq \lambda(D)$.*

Similarly as in the previous case, one interesting question is if we require optimality on the coloring rather than on the path, can we obtain a similar result?

Question 2. *Let D be a digraph and let $\mathcal{S}$ be a minimum coloring of D. Is it true that D contains a path P orthogonal to $\mathcal{S}$?*

[Berge 1982a] observed that the answer to this question is also negative. See Figure 1b. Note that $\{\{v_1, v_4\}, \{v_2\}, \{v_3, v_4\}\}$ is a minimum coloring of the digraph D depicted, but there is no path orthogonal to it.

Motivated by the previous observations and by his study on perfect graphs, [Berge 1982a] introduced two classes of digraphs, named α-diperfect and χ-diperfect digraph. A digraph D is *α-diperfect* if for every maximum stable set S of D, there is a path partition orthogonal to S and this property holds for every induced subdigraph of D. A digraph D is *χ-diperfect* if for every minimum coloring $\mathcal{S}$ of D, there is a path P orthogonal to $\mathcal{S}$, and this property holds for every induced subdigraph of D. The ultimate goal for the classes of α-diperfect digraph and χ-diperfect digraphs is to obtain a characterization of these classes in terms of forbidden induced subdigraphs. Some advance has been made towards this goal [de Paula Silva et al. 2022a, de Paula Silva et al. 2022b, Freitas and Lee 2022a, Freitas and Lee 2022b, Sambinelli et al. 2022, de Paula Silva et al. 2025], and it seems that this is a very hard problem. Many examples of minimal non-α-diperfect digraphs and minimal non-χ-diperfect digraphs have been discovered and they do not seem to have an evident pattern [de Paula Silva et al. 2023, de Paula Silva et al. 2023, de Paula Silva et al. 2024].

In 1982, [Neumann-Lara 1982] introduced a directed analogue of the usual concept of coloring in a digraph. In this scenario, stable sets are replaced by (induced) acyclic digraphs. A digraph is *acyclic* if it does not contain any directed cycle. Let D be a digraph. A *dicoloring* of a digraph is a partition $S_1, \ldots, S_t$ of $V(D)$ such that, for every $i \in \{1, \ldots, t\}$, $D[S_i]$ is an acyclic digraph. The minimum t for which D admits a dicoloring of size t is called the *dichromatic number* of D and it is denoted by $\vec{\chi}(D)$. In the early 2000s, this notion was reintroduced by [Mohar 2003, Bokal et al. 2004] and since then many results analogous to classical results in graph theory have been proven using the notion of dicoloring [Aboulker et al. 2021, Picasarri-Arrieta 2024b, Kawarabayashi and Picasarri-Arrieta 2025]. We refer the reader to the PhD thesis of [Aubian 2023] and [Picasarri-Arrieta 2024a] for a more complete overview of the topic.

Since every stable set corresponds to an acyclic digraph, Theorems 1 and 2 are trivially true if we replace stable sets by induced acyclic subdigraph and, coloring by dicoloring, respectively. It is natural to ask, then, what happens if we make the same change in Questions 1 and 2.

Question 3. *Let D be a digraph and let T be a maximum induced acyclic subdigraph of D. Is it true that D contains a path partition orthogonal to T?*

Question 4. *Let D be a digraph and let $\mathcal{S}$ be a minimum dicoloring of D. Is it true that D contains a path P orthogonal to $\mathcal{S}$?*

It turns out that this change makes both problems much simpler. In this paper, we give affirmative answers for both questions.

2. Answering Questions 3 and 4

Before we present our results, we need some auxiliary concepts. Let G be a graph, let $X \subseteq V(G)$, and let M be a matching of G. We say that M *covers* X if every vertex of X is incident to some edge of M. If $S \subseteq V(G)$, we denote by $N_G(S)$ the set of neighbors of S in G. When G is clear from the context, we write simply $N(S)$. The following theorem is a classical result in Graph Theory, due to [Hall 1987].

Theorem 3 (Hall's Theorem)**.** *A bipartite graph $G[X, Y]$ has a matching that covers X if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$.*

If D is a digraph and $X_1, X_2 \subseteq V(D)$ are disjoint sets of D, then we denote by $D[X_1, X_2]$ the bipartite subdigraph of D with vertex-set $X_1 \cup X_2$ and arc-set containing all arcs of D between X_1 and X_2. A *matching* of a digraph is a matching of its underlying graph. We say that $M \subseteq A(D)$ is a *directed matching* in $D[X_1, X_2]$ if M is a matching in $U(D[X_1, X_2])$ and all arcs of M are directed from X_1 to X_2. Moreover, we say that M *covers* X_1 (respec., *covers* X_2) if M covers X_1 (respec., covers X_2) in $U(D[X_1, X_2])$.

Lemma 1. *Let D be a digraph and let T_1 be a maximum induced acyclic subdigraph of D. If T_2 is an induced acyclic subdigraph of $D - V(T_1)$, then there exists a directed matching covering $V(T_2)$ in $D[V(T_1), V(T_2)]$.*

Proof. Let $X_1 = V(T_1)$ and let $X_2 = V(T_2)$. Let D' be the subdigraph of D with vertex set $X_1 \cup X_2$ and containing only the arcs of D that are directed from X_1 to X_2. Let $G = G[X_1, X_2] = U(D')$. Note that if $G[X_1, X_2]$ has a matching that covers X_2, then there is a directed matching in $D[X_1, X_2]$ that covers X_2 and the result follows. Towards a contradiction, assume that such a matching does not exist. Then, by Hall's Theorem, there exists a subset $S \subseteq X_2$ such that $|S| > |N_G(S)|$. Also note that $S \cup (X_1 \setminus N_G(S))$ induces an acyclic subdigraph of D. However, $|S \cup (X_1 \setminus N_G(S))| > |X_1| = |V(T_1)|$, a contradiction. $\square$

Now we prove Theorem 4 which gives an affirmative answer to Question 3.

Theorem 4. *Let D be a digraph and let T be a maximum induced acyclic subdigraph of D. There exists a path partition $\mathcal{P}$ of D such that ini$(\mathcal{P}) = V(T)$.*

Proof. The proof follows by induction on $|V(D)|$. If $T = D$, then the path partition consisting of $|V(D)|$ trivial paths satisfies the desired property. So we may assume that $T \neq D$. Let T' be a maximum induced acyclic subdigraph of $D - V(T)$. By induction hypothesis, there exists a path partition $\mathcal{P}'$ of $D - V(T)$ such that ini$(\mathcal{P}') = V(T')$. By Lemma 1, there exists a directed matching M covering $V(T')$ in $D[V(T), V(T')]$. Let $\mathcal{Q}$ be the set of trivial paths consisting of every vertex of T that is not covered by M. Let $\mathcal{R}$ be the set of disjoint paths obtained from $\mathcal{P}'$ by extending each path $P \in \mathcal{P}'$ with the arc of M incident to ini(P). Then $\mathcal{P} = \mathcal{R} \cup \mathcal{Q}$ is a path partition orthogonal to T and ini$(\mathcal{P}) = V(T)$. This finishes the proof. $\square$

Now we prove Theorem 5 which gives an affirmative answer to Question 4.

Theorem 5. *Let D be a digraph and let $\mathcal{S}$ be a minimum dicoloring of D. Then there exists a path P of D orthogonal to $\mathcal{S}$.*

Proof. Let $t = \vec{\chi}(D)$ and let $\mathcal{S} = \{S_1, \ldots, S_t\}$ be a minimum dicoloring of D. Let $F = \{(u,v) \in A(D) : u \in S_i, v \in S_j, i < j\}$. Note that since each $D[S_i]$ is acyclic, $D - F$ is also acyclic. Consider the subdigraph $D' = (V(D), F)$. Suppose first that D' has a path P of order t. Then clearly P is a path orthogonal to $\mathcal{S}$. So we may assume that the longest path of D' has order at most $t - 1$. By Gallai-Hasse-Roy-Vitaver Theorem, $\chi(D') \leq t-1$. Let $\mathcal{C} = \{C_1, \ldots, C_{\chi(D')}\}$ be a minimum coloring of D'. Since each color class C_j is a stable set in D', the arc set of $D[C_j]$ contains no arc of F and, so $D[C_j]$ is a subdigraph of $D - F$. This implies that $D[C_j]$ is acyclic. Hence, $\{C_1, \ldots, C_{\chi(D')}\}$ is a dicoloring of D of size smaller than t, a contradiction. $\square$

[Linial 1981] proposed two generalizations of Gallai-Milgram Theorem and Gallai-Hasse-Roy-Vitaver Theorem that use a positive integer k as a measure of optimality for the path partition and the coloring, respectively. These generalizations have led to two conjectures that remain open [Berge 1982b, Hartman 2006, Sambinelli 2018]. Similarly as above, if we replace in the definition stable sets by induced acyclic digraphs, we can use the same the ideas described above to prove versions of these two conjectures in this setting (see [de Paula Silva et al. 2026] for a complete version of our results.)

References

Aboulker, P., Charbit, P., and Naserasr, R. (2021). Extension of Gyárfás-Sumner conjecture to digraphs. *The Electronic Journal of Combinatorics*.

Aubian, G. (2023). *Colouring digraphs*. PhD thesis, Université Paris Cité.

Bang Jensen, J. and Gutin, G. Z. (2008). *Digraphs: theory, algorithms and applications*. Springer Science & Business Media.

Berge, C. (1982a). Diperfect graphs. *Combinatorica*, 2(3):213–222.

Berge, C. (1982b). k-optimal partitions of a directed graph. *European Journal of Combinatorics*, 3(2):97–101.

Bokal, D., Fijavz, G., Juvan, M., Kayll, P. M., and Mohar, B. (2004). The circular chromatic number of a digraph. *Journal of Graph Theory*, 46(3):227–240.

Bondy, J. A. and Murty, U. S. R. (2008). *Graph Theory*. Springer.

de Paula Silva, C. A., da Silva, C. N., and Lee, O. (2023). Obstructions for χ-diperfectness. In *Procedia Computer Science. LAGOS 2023: XII Latin-American Algorithms, Graphs and Optimization Symposium. To appear.*

de Paula Silva, C. A., da Silva, C. N., and Lee, O. (2024). BE-Diperfect Digraphs with Stability Number Two. *The Electronic Journal of Combinatorics*, pages P2–43.

de Paula Silva, C. A., da Silva, C. N., and Lee, O. (2025). Acyclic α-diperfect digraphs with stability number two. *Procedia Computer Science*, 273:110–116.

de Paula Silva, C. A., Nunes da Silva, C., and Lee, O. (2022a). χ-Diperfect digraphs. *Discrete Mathematics*, 345(9):112941.

de Paula Silva, C. A., Nunes da Silva, C., and Lee, O. (2022b). On χ-Diperfect Digraphs with Stability Number Two. In Castañeda, A. and Rodríguez-Henríquez, F., editors, *LATIN 2022: Theoretical Informatics*, pages 460–475, Cham. Springer International Publishing.

de Paula Silva, C. A., Nunes da Silva, C., and Lee, O. (2023). A family of counterexamples for a conjecture of Berge on α-diperfect digraphs. *Discrete Mathematics*, 346(8):113458.

de Paula Silva, C. A., Nunes da Silva, C., and Lee, O. (2026). Orthogonality between acyclic subdigraphs and paths in digraphs. *arXiv preprint arXiv:2603.17115*. `arXiv:2603.17115`.

Freitas, L. I. B. and Lee, O. (2022a). 3-anti-circulant digraphs are α-diperfect and BE-diperfect. *Open Journal of Discrete Mathematics*.

Freitas, L. I. B. and Lee, O. (2022b). Some Results on Berge's Conjecture and Begin–End Conjecture. *Graphs and Combinatorics*, 38(4):1–23.

Gallai, T. (1968). On directed paths and circuits. *Theory of graphs*, 38:2054.

Gallai, T. and Milgram, A. N. (1960). Verallgemeinerung eines graphentheoretischen Satzes von Rédei. *Acta Sci Math*, 21:181–186.

Hall, P. (1987). On representatives of subsets. In *Classic Papers in Combinatorics*, pages 58–62. Springer.

Hartman, I. B.-A. (2006). Berge's conjecture on directed path partitions—a survey. *Discrete mathematics*, 306(19-20):2498–2514.

Hasse, M. (1965). Zur algebraischen Btegründung der Graphentheorie. i. *Mathematische Nachrichten*, 28(5-6):275–290.

Kawarabayashi, K.-i. and Picasarri-Arrieta, L. (2025). An analogue of Reed's conjecture for digraphs. In *Proceedings of the 2025 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 3310–3324. SIAM.

Linial, N. (1981). Extending the Greene-Kleitman theorem to directed graphs. *Journal of Combinatorial Theory, Series A*, 30(3):331–334.

Mohar, B. (2003). Circular colorings of edge-weighted graphs. *Journal of Graph Theory*, 43(2):107–116.

Neumann-Lara, V. (1982). The dichromatic number of a digraph. *Journal of Combinatorial Theory, Series B*, 33(3):265–270.

Picasarri-Arrieta, L. (2024a). *Digraph colouring*. PhD thesis, Université Côte d'Azur.

Picasarri-Arrieta, L. (2024b). Strengthening the Directed Brooks' Theorem for oriented graphs and consequences on digraph redicolouring. *Journal of Graph Theory*, 106(1):5–22.

Roy, B. (1967). Nombre chromatique et plus longs chemins d'un graphe. *Revue française d'informatique et de recherche opérationnelle*, 1(5):129–132.

Sambinelli, M. (2018). *Partition problems in graphs and digraphs*. PhD thesis, State University of Campinas - UNICAMP.

Sambinelli, M., da Silva, C. N., and Lee, O. (2022). α-Diperfect digraphs. *Discrete Mathematics*, 345(5):112759.

Vitaver, L. (1962). Determination of minimal coloring of vertices of a graph by means of boolean powers of the incidence matrix. In *Doklady Akademii Nauk*, volume 147, pages 758–759. Russian Academy of Sciences.