# $p-$Role Assignment for Cayley Graph $H_{\ell,p}$

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***Abstract.*** *We study the role assignment problem in the family of Cayley graphs $H_{\ell,p}$. We show that there exists a natural graph homomorphism from $H_{\ell,p}$ to a circulant graph $\Gamma$, which implies that every graph in this family admits a $p$-role assignment. We also present an efficient algorithm to compute such assignments in $O(\ell p^{\ell-1})$ time. Our results highlight the interplay between algebraic structure and role assignments in highly symmetric graphs.*

## 1. Introduction

The study of algorithmic problems in structured graph classes is a central theme in graph theory and theoretical computer science. Many problems that are computationally hard in general graphs become tractable or reveal interesting structural properties when restricted to specific graph families. The graphs $H_{\ell,p}$, originally introduced by Holyer [1981] and later studied in the context of interconnection networks by Ribeiro et al. [2014], form a highly symmetric family of Cayley graphs with desirable properties such as regularity, vertex-transitivity and relatively small diameter.

The role assignment, introduced by Everett and Borgatti [1991], captures the notion of structural equivalence in networks by mapping vertices to a set of roles, such that vertices assigned with the same role share the same neighborhood role set. Investigating role assignment in highly symmetric graph classes, such graphs may provide insights both into the structural properties of these graphs and into the design and analysis of communication patterns in interconnection networks.

In this work, we investig ate the role assignment problem in the of family $H_{\ell,p}$. We establish a connection between these graphs and a graph defined in ($\mathbb{Z}_p$). In particular, we show that there exists a natural graph homomorphism from $H_{\ell,p}$ to a graph ($\Gamma$) generated by the set ($\{\pm 1, \pm 2, \ldots, \pm(l-1)\}$). As a consequence, we prove that every graph $H_{\ell,p}$ admits a $p$-role assignment. This result provides a simple algebraic construction of role assignment for this class of Cayley graphs and highlights the interplay between the combinatorial structure of $H_{\ell,p}$ and homomorphisms into circulant graphs.

## 2. Basic Preliminaries

In this section, we introduce the main definitions of role assignment in graphs and the family of graph $H_{\ell,p}$.

### 2.1. Role Assignments

We begin by recalling the fundamental properties of role assignments, following the conventions established in Mesquita [2022].

**Definition 1** (Everett and Borgatti [1991]). Let $G$ and $R$ be graphs. A mapping $\phi : V(G) \to V(R)$ is called a *role assignment* if for every vertex $u \in V(G)$ we have $\phi(N_G(u)) = N_R(\phi(u))$.

**Definition 2.** Let $G$ be a graph and let $r$ be a positive integer. We say that $G$ admits an $r$-*role assignment* if there exists a surjective mapping $\phi : V(G) \to \{1, \ldots, r\}$ that is a role assignment.

**Definition 3.** Let $G$ and $R$ be graphs. We say that $G$ admits an $R$-*role assignment* if there exists a role assignment $\phi : V(G) \to V(R)$.

To avoid notation conflict with the parameter $p$ of the graph $H_{l,p}$, we denote the role assignment function by $\phi : V(G) \to V(R)$ (conventionally denoted by $p$ in Mesquita [2022]).

### 2.2. The family of graphs $H_{\ell,p}$

The graph $H_{\ell,p}$ is defined as follows. The vertex set consists of all $l$-tuples $(x_1, \ldots, x_l)$ with $x_i \in \{0, 1, \ldots, p-1\}$ such that $\sum_{i=1}^{l} x_i \equiv 0 \pmod{p}$, where $p \in \mathbb{Z}_+$.

Two vertices are adjacent if and only if there exists a pair of corresponding coordinates whose values differ by exactly one unit, with one coordinate increased by 1 and the other decreased by 1.

**Definition 4** (Holyer [1981]). For $l \geq 3$ and $p \geq 3$, let $H_{l,p} = (V_{l,p}, E_{l,p})$ be the graph defined as follows:

- $V_{l,p} = \{x = (x_1, \ldots, x_l) \in \mathbb{Z}_p^l : \sum_{i=1}^{l} x_i \equiv 0 \pmod{p}\}$;
- $E_{l,p} = \{xy :$ there are distinct $i, j$ such that $y_k \equiv_p x_k$ for $k \neq i, j$ and $y_i \equiv_p x_i + 1, y_j \equiv_p x_j - 1\}$.

Examples of the graph $H_{\ell,p}$ are the graphs $H_{3,4}$ and $H_{4,p}$ illustrated in Figures 1 and 2 respectively.

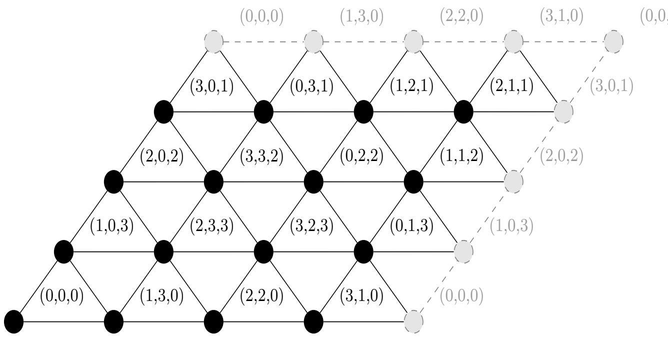


**Figure 1. A drawing of $H_{3,4}$.**

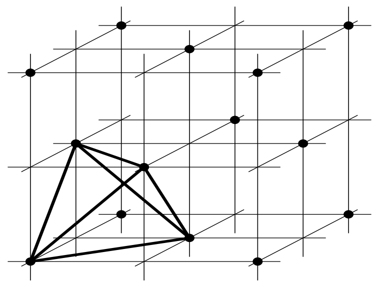

**Figure 2. A generic example of $H_{4,p}$ graph. Extracted from Holyer [1981]**

## 3. A $p$-role assignment in graphs $H_{\ell,p}$

In this section, we present an algorithm for applying a $p$-role assignment in graphs $H_{\ell,p}$.

**Definition 5.** For integers $l \geq 3$ and $p \geq 3$, define the circulant graph $\Gamma = (\mathbb{Z}_p, F_{l,p})$ as follows.

The vertex set is $\mathbb{Z}_p$, and two vertices $a, b \in \mathbb{Z}_p$ are adjacent if and only if $b-a \in S_{l,p}$, where $S_{l,p} = \{\pm 1, \pm 2, \ldots, \pm(l-1)\} \pmod p$.

**Theorem 1.** *For every integer $l \geq 3$ and $p \geq 3$, there exists a homomorphism $\phi : H_{l,p} \to \Gamma$, and hence $H_{\ell,p}$ admits a $p$-role assignment.*

*Proof.* Define the function $\phi : H_{\ell,p} \to \Gamma$ by

$$\phi(x_1, x_2, \cdots, x_l) = \sum_{k=1}^{l} k \cdot x_k \pmod p$$

Let $xy$ be an edge of $H_{\ell,p}$. By definition of adjacency in $H_{\ell,p}$, the vertex $x$ is obtained from $y$ by increasing the $i$-th coordinate by 1 and decreasing the $j$-th coordinate by 1, for some $i \neq j$, and conversely. Hence, the difference $\phi(x) - \phi(y)$ is given by:

$$\phi(x) - \phi(y) = (k_i \cdot (x_i + 1) + k_j \cdot (x_j - 1)) - (k_i x_i + k_j x_j) = k_i - k_j$$

Taking the weights $k = \{1, 2, \ldots, \ell\}$, it follows that the difference is always of the form $i - j \pmod p$. Since $i \neq j$, we have $\Delta = i - j \in \{\pm 1, \pm 2, \ldots, \pm(\ell - 1)\} \pmod p$.

To establish the reverse inclusion. Let $x \in V(H_{\ell,p})$ and let $z \in N_\Gamma(\phi(x))$. By the definition of the circulant graph $\Gamma$, we have $z \equiv \phi(x) + \Delta \pmod p$, where $\Delta \in \{\pm 1, \pm 2, \ldots, \pm(\ell - 1)\}$. We must find a vertex $y \in N_{H_{\ell,p}}(x)$ such that $\phi(y) = z$.

Let $y$ be a neighbor obtained from $x$ by increasing the $i$-th coordinate by 1 and decreasing the $j$-th coordinate by 1, for some distinct $i, j \in \{1, 2, \ldots, l\}$. As established, $\phi(y) - \phi(x) \equiv i - j \pmod p$. We may choose the indices $i$ and $j$ so that $i - j = \Delta$ as follows:

If $\Delta > 0$, choose $i = \Delta + 1$ and $j = 1$. Since $1 \leq \Delta \leq \ell - 1$, it follows that $2 \leq i \leq \ell$. If $\Delta < 0$, we may write $\Delta = -d$ for some positive integer $d$ with $1 \leq d \leq \ell - 1$. Choose $i = 1$ and $j = d + 1$. Thus, $2 \leq j \leq \ell$.

In both cases, $i$ and $j$ are valid and distinct indices in $\{1, 2, \ldots, \ell\}$. Therefore, the required neighbor $y$ exists in $H_{l,p}$ and satisfies $\phi(y) = z$. This proves that $\phi(N_{H_{\ell,p}}(x)) = N_\Gamma(\phi(x))$.

□

### 3.1. $(p, l)$-Gray Code with role assignment

To apply Theorem 1 and generate the vertices of $H_{\ell,p}$, we use a modified version of the $(p, l)$-Gray code shown in Ribeiro et al. [2010]. The Gray code provides an ordering of vectors where consecutive elements differ in only one coordinate, which is closely related to the construction of Hamiltonian cycles in $H_{\ell,p}$.

The algorithm enumerates all vertices of $H_{\ell,p}$ and their roles in $O(\ell \cdot p^{\ell-1})$ total time ($O(\ell)$ per vertex) by iterating from 0 to $p^{\ell-1} - 1$. Each integer is converted to base $p$ and transformed via Gray code to obtain the first $\ell - 1$ coordinates. The last coordinate is chosen to satisfy $\sum_{i=1}^{\ell} x_i \equiv 0 \pmod{p}$, ensuring the vector belongs to $V_{\ell,p}$. Finally, the algorithm computes $\phi(x)$ to assign a role in $\mathbb{Z}_p$ to each vertex.

To illustrate the algorithm, consider the graph $H_{3,6}$ in Figure 3[cite: 76, 78]. Each vertex $x = (x_1, x_2, x_3) \in \mathbb{Z}_6^3$ satisfies $x_1 + x_2 + x_3 \equiv 0 \pmod{6}$ and is annotated with its coordinates and assigned role $\phi(x)$, demonstrating how the procedure computes both simultaneously.

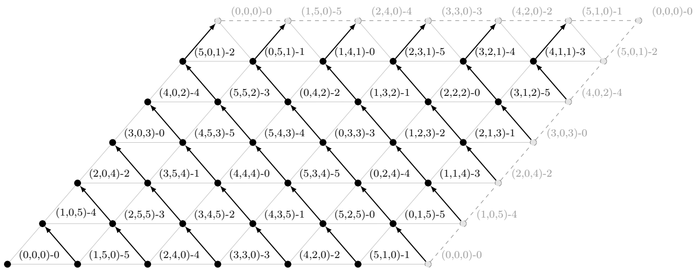


**Figure 3. A labeled instance of $H_{3,6}$ with the values of $\phi(x)$ for each vertex.**

From this construction, the vertices are partitioned into roles in $\mathbb{Z}_6$, and the induced role graph corresponds to a cycle, as shown in Figure 6.

We also consider smaller instances for illustration. For $H_{3,4}$, the algorithm produces four distinct roles, resulting in the complete graph $K_4$. For $H_{4,3}$, it produces three roles, yielding the cycle $C_3$, as shown in Figure 4 and Figure 5, respectively.

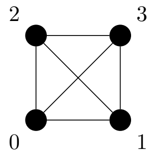


**Figure 4. A $\Gamma$ 4-assignment for** $H_{3,4}$

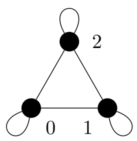


**Figure 5. A $\Gamma$ 3-assignment for** $H_{4,3}$

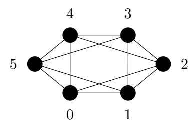


**Figure 6. A $\Gamma$ 6-assignment for $H_{3,6}$**

## 4. Conclusion

In this work, we studied the role assignment problem for the family of Cayley graphs $H_{\ell,p}$. We showed that there exists a natural homomorphism from $H_{\ell,p}$ to the circulant graph $\Gamma$, which implies that every graph in this family admits a $p$-role assignment.

Based on this result, we presented an efficient algorithm to compute the role associated with each vertex by evaluating the function $\phi(x) = \sum_{k=1}^{\ell} k \cdot x_k \pmod{p}$, running in $O(\ell p^{\ell-1})$ time. This provides a practical method for constructing role assignments in these graphs.

Finally, we illustrated the construction with small examples, namely the graphs $H_{3,4}$ and $H_{4,3}$, whose corresponding role graphs are $K_4$ and $C_3$, respectively. These examples highlight the homomorphic mapping and reinforce the connection between the algebraic structure of $H_{\ell,p}$ and role assignments.

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