A proof for Berge's Dual Conjecture for Bipartite Digraphs

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Abstract. *Given a (vertex)-coloring $\mathcal{C} = \{C_1, C_2, ...C_m\}$ of a digraph D and a positive integer k, the k-norm of C is defined as $|\mathcal{C}|_k = \sum_{i=1}^{m} min\{|C_i|, k\}$. A coloring $\mathcal{C}$ is k-optimal if its k-norm $|\mathcal{C}|_k$ is minimum over all colorings. A (path) k-pack $\mathcal{P}^k$ is a collection of at most k vertex-disjoint paths. A coloring $\mathcal{C}$ and a k-pack $\mathcal{P}^k$ are orthogonal if each color class intersects as many paths as possible in $\mathcal{P}^k$, that is, if $|C_i| \geq k$, $|C_i \cap P_j| = 1$ for every path $P_j \in \mathcal{P}^k$, otherwise each vertex of C_i lies in a different path of $\mathcal{P}^k$. In 1982, Berge conjectured that for every k-optimal coloring $\mathcal{C}$ there is a k-pack $\mathcal{P}^k$ orthogonal to $\mathcal{C}$. This conjecture is false for arbitrary digraphs, having a counterexample with odd cycle. In this paper we prove this conjecture for bipartite digraphs.*

1. Introduction

Given a digraph D, we denote its set of vertices by $V(D)$ and its set of arcs by $A(D)$. The *order* of D is the cardinality of $V(D)$, denoted by n. An *independent (vertex)-set* is a set of pairwise non-adjacent vertices. We denote as $\alpha(D)$ the cardinality of a maximum independent vertex-set of a digraph D. A *vertex cover* of a digraph D is a vertex cover of its underlying undirected graph $G(D)$, i. e., a subset V' of $V(G)$ such that every edge $e \in E(G)$ has at least one end in V'. We denote the cardinality of the minimum vertex cover as $\tau(D)$. Gallai [Gallai 1959] observed a relation between a maximum independent set and a minimum vertex cover for general graphs.
Theorem 1 (Gallai)**.** *For any graph G, $\alpha(G) + \tau(G) = n$.*

A digraph D is *(X, Y)-bipartite* if $V(D) = X \cup Y$, $X \cap Y = \emptyset$ and $D[X]$ and $D[Y]$ are independent sets. A *matching* in a digraph D is a subset $M \in A(D)$ where, for every vertex $v \in V$, at most one arc of M is incident with v. We denote the cardinality of a maximum matching by $\alpha'(D)$. König [König 1931] showed the equivalence between the maximum matching problem and the minimum vertex cover problem in bipartite graphs:
Theorem 2 (König)**.** *If G is a bipartite graph, then $\tau(G) = \alpha'(G)$.*
Corollary 1. *If G is a bipartite graph then $\alpha'(G) + \alpha(G) = n$.*

A *path* $(v_1, v_2, \ldots, v_p)$ is a sequence of vertices such that arc $(v_i, v_{i+1}) \in A(D)$ for $1 \leq i < p$. The *cardinality of a path* is the number of vertices p in the sequence, which is an unusual definition. We denote by $\lambda(D)$ the cardinality of a maximum path in D. An *m-(vertex)-coloring* (or simply a *coloring*) $\mathcal{C} = \{C_1, C_2, ..., C_m\}$ is a collection

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of disjoint independent sets, also called color classes, where $\bigcup\{C_i : C_i \in \mathcal{C}\} = V(G)$. A coloring $\mathcal{C}$ of D is optimal if there is no other coloring with few color classes. The chromatic number of D, denoted by $\chi(D)$, is equal to the size of an optimal coloring of D. Roy [Roy 1967] and Gallai [Gallai 1968], independently proved that the size of an optimal coloring is smaller or equal to the size of the largest path in D. Shortly after Gallai-Roy's theorem, Mirsky [Mirsky 1971] proved the equality for a specific class of digraphs.

Theorem 3 (Gallai-Roy)**.** *For every digraph D, $\chi(D) \leq \lambda(D)$.*

Theorem 4 (Mirsky)**.** *If D is an acyclic transitive digraph, then $\lambda(D) = \chi(D)$.*

Some time later, a new metric of minimality and maximality was established, using a positive integer k to redefine optimal collections of paths and optimal colorings. Given a coloring $\mathcal{C} = \{C_1, C_2, ..., C_m\}$ and a positive integer k, the *k-norm* of $\mathcal{C}$ is defined as

$$|\mathcal{C}|_k = \sum_{i=1}^{m} min\{|C_i|, k\}.$$

A coloring $\mathcal{C}$ of D is *k-optimal* if there is no coloring $\mathcal{B}$ of D where $|\mathcal{B}|_k < |\mathcal{C}|_k$. We denote as $\chi_k(D)$ the k-norm of a k-optimal coloring of D. A *(path) k-pack* $\mathcal{P}^k$ is a collection of at most k (vertex-)disjoint paths of a digraph D. The *weight* of $\mathcal{P}^k$ is defined as $| \cup_{P\in\mathcal{P}^k} V(P)|$ and denoted as $||\mathcal{P}_k||$. A k-pack is *optimal* if its weight is maximum. The weight of an optimal k-pack for D is denoted as $\lambda_k(D)$. Using this new metric, Greene [Greene 1976] showed that the equality holds for acyclic transitive digraphs, and Linial [Linial 1981] conjectured that an inequality should hold in general:

Theorem 5 (Greene)**.** *If D is an acyclic transitive digraph and k a positive integer, then $\chi_k(D) = \lambda_k(D)$.*

Conjecture 1 (Linial's Dual Conjecture)**.** *If D is a digraph and k is a positive integer, then $\chi_k(D) \leq \lambda_k(D)$.*

A stronger conjecture was proposed introducing the concept of orthogonality. A coloring $\mathcal{C} = \{C_1, C_2, ..., C_m\}$ and a k-pack are *orthogonal* if each color class $C_i \in \mathcal{C}$ meets $\min\{|C_i|, k\}$ different paths of $\mathcal{P}_k$.

Conjecture 2 (Berge's Dual Conjecture - *false*)**.** *For every k-optimal coloring $\mathcal{C}$ of a digraph D, there is a k-pack orthogonal to $\mathcal{C}$.*

Berge's Dual Conjecture implies Linial's Dual Conjecture because $\lambda_k(D)$ is at least the size of the set of vertices covered by the k-pack orthogonal to the coloring, which in turn is exactly $\chi_k(D)$. However, Berge's Dual Conjecture is known to be false for arbitrary digraphs, having a counterexample, which is depicted in Figure 1. The digraph has an 1-optimal coloring, but there is no path able to meet every color of this digraph.

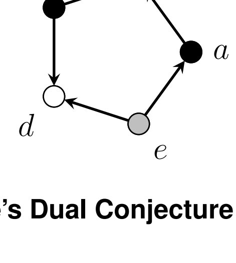

Figure 1. Berge's Dual Conjecture Counterexample

On the other hand, it was not known so far whether the conjecture was true for bipartite digraphs. Note that the underlying undirected graph of the counterexample is an odd cycle, and bipartite digraphs are free of such structure.

There is an analogue set of theorems and conjectures which are obtained by changing the roles of paths and independent sets. Actually, in historical terms the version of such theorems and conjectures first discovered where not the ones presented in this paper, but the ones with roles changed. That is why Berges's Dual Conjecture and Linial's Dual Conjecture are called *Dual* conjectures in this paper. Moreover, the validity of the primal version of the Berge's Conjecuture for the bipartite digraphs has already been shown [Berge 1982] by Berge himself in the same paper where the primal conjecture was proposed. A comprehensive survey on both primal and dual versions of these problems can be found in [Hartman 2006].

2. Berge Dual Conjecture for Bipartite Digraphs

In this section, we show that for bipartite digraphs the problem of finding a k-pack orthogonal to the given optimal coloring reduces to the problem of finding a k-pack composed by a selection of $\min\{\alpha'(D), k\}$ arcs of a maximum matching of D together with $k - \alpha'(D)$ unmatched vertices of D which must be selected with some criteria. This implies that, in the bipartite case, the directions of the arcs are not relevant to the proof of Berge's Dual Conjecture. Moreover, the fact of the digraphs are bipartite, and, thus, admit a 2-coloring, implies the important observation that for every coloring $\mathcal{C}$ the k-norm of C is at most $2k$, so $\chi_k(D) \leq 2k$. Therefore, if there is two color classes of size at least k, then there cannot be more color classes.

Theorem 6. *Let D be a bipartite digraph and $\mathcal{C}$ a k-optimal coloring of D. Then, there is a k-pack $\mathcal{P}$ orthogonal to $\mathcal{C}$.*

Proof. Let $\mathcal{C} = \{C_1, C_2, ..., C_m\}$ be a k-optimal coloring of D. We may assume, without loss of generality, that $|C_1| \geq |C_2| \geq ... \geq |C_m|$. We will prove that there always is a k-pack $\mathcal{P}$ composed by a matching and unmatched vertices that meets $\min\{|C_i|, k\}$ vertices for each C_i. If $\mathcal{P}$ meets $\min\{|C_i|, k\}$ vertices for every $C_i \in \mathcal{C}$ the only missing restriction to claim that orthogonality holds is the assurance that each color class meets different paths. But since an arc in a matching can only connect vertices from different color classes and unmatched vertices form different trivial paths, this condition is immediately satisfied. Thus, $\mathcal{P}$ is a k-pack orthogonal to $\mathcal{C}$ whenever it has the described structure.

Consider first the case in which $\alpha(D) \leq k$. In this case, $\min\{|C_i|, k\} = |C_i|$ for each C_i. Therefore, every vertex must be in some path of an orthogonal k-pack. Let M be a maximum matching of D and let I be the set of vertices of D not matched by M. If $\alpha'(D) + |I| \leq k$, the arcs in M together with the trivial paths with vertices of I form the k-pack we are seeking. But this inequality is always valid since $n = 2 * \alpha'(D) + |I|$ and, by Corollary 1,

$$\begin{aligned}
\alpha'(D) + \alpha(D) &= 2 * \alpha'(D) + |I| \\
\alpha(D) &= \alpha'(D) + |I| \\
&\leq k.
\end{aligned}$$

We may thus assume that $\alpha(D) > k$. If $|C_1| \leq k$, then $\min\{|C_i|, k\} = |C_i|$ for each C_i and $|\mathcal{C}|_k = V(D)$. In this case, an arbitrary coloring having a set of size $\alpha(D)$ as a color class has smaller k-norm; a contradiction. Thus, $|C_1| > k$. Consider first the case in which $n - |C_1| \leq k$. We will now argue that $|C_1| = \alpha(D)$ in this case. Assume the contrary, that is, $|C_1| < \alpha(D)$. Now let $\mathcal{C}' = \{C'_1, C'_2, ..., C'_{m'}\}$ be some coloring of D such that $|C'_1| = \alpha(D)$. The k-norm of $\mathcal{C}'$ is thus at most $k + n - \alpha(D) < k + n - |C_1|$, contradicting the fact that $\mathcal{C}$ is k-optimal. So, $|C_1| = \alpha(D)$. By Corollary 1, $\alpha'(D) = n - \alpha(D) = n - |C_1| \leq k$. One maximum matching M with $\alpha'(D)$ arcs cannot have an arc with both ends in $V(D) - C_1$ since that would imply that M would also have an arc with both ends in C_1, which is a stable set. Thus, every maximum matching M has precisely one end in C_1 and the other end in $V(D) - C_1$. Since $\alpha'(D) \leq k$ and every color class C_i, $i \neq 1$ has size at most k, a k-pack $\mathcal{P}$ composed by some maximum matching M together with $k - \alpha'(D)$ vertices of C_1 as trivial paths is orthogonal to $\mathcal{C}$.

Finally, consider the case in which $n - |C_1| > k$. Since $\chi_k(D) \leq 2k$, we deduce that $\mathcal{C}$ has precisely two color classes, both of size larger than k and $\chi_k(D) = 2k$. Then, an arbitrary subset of k arcs of some maximum matching M form a k-pack orthogonal to $\mathcal{C}$. It suffices thus to show that $\alpha'(D) \geq k$. By Corollary 1, $\alpha'(D) = n - \alpha(D)$ and if $n - \alpha(D) < k$, then we may define a coloring $\mathcal{C}' = \{C'_1, C'_2, ..., C'_{m'}\}$ of D such that $|C'_1| = \alpha(D)$ and whose k-norm would be smaller than $2k$, a contradiction. Thus $\alpha'(D) \geq k$ indeed. And in every case considered there is a k-pack $\mathcal{P}$ orthogonal to $\mathcal{C}$.

□

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