On Linial's Conjecture for Split Digraphs

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Abstract. *In this paper we show that Linial's Conjecture holds for two classes of split digraphs, namely the spider digraphs and the k-loose digraphs.*

1. Introduction

The digraphs considered in this text do not contain loops or parallel arcs and by path we mean directed path. Let D be a digraph. We denote by $V(P)$ the set of vertices of a path P. The **size** of a path P, denoted by $|P|$, is $|V(P)|$[1]. We denote by $\lambda(D)$ the size of the longest path in D and by $\alpha(D)$ the size of a maximum stable set. A *path partition* $\mathcal{P}$ of D is a set of vertex-disjoint paths of D that cover $V(P)$. We say that $\mathcal{P}$ is an **optimal** path partition if there is no path partition $\mathcal{P}'$ of D such that $|\mathcal{P}'| < |\mathcal{P}|$. We denote by $\pi(D)$ the size of an optimal path partition of a digraph D.

Dilworth [Dilworth 1950] showed that for every transitive acyclic digraph D we have $\pi(D) = \alpha(D)$. Note that this equality is not valid for any digraph; for example, if D is a directed cycle with 5 vertices, then $\pi(D) = 1$ and $\alpha(D) = 2$. However, Gallai and Milgram [Gallai and Milgram 1960] have shown that $\pi(D) \leq \alpha(D)$ for every digraph D.

Greene and Kleitman [Greene and Kleitman 1976] proved a generalization of Dilworth's Theorem described next. Let k be a positive integer. The ***k*-norm** of a path partition $\mathcal{P}$, denoted by $|\mathcal{P}|_k$, is defined as $|\mathcal{P}|_k = \sum_{P \in \mathcal{P}} \min\{|P|, k\}$. We say that $\mathcal{P}$ is a ***k*-optimal path partition** if there is no path partition $\mathcal{P}'$ such that $|\mathcal{P}'|_k < |\mathcal{P}|_k$. We denote by $\pi_k(D)$ the k-norm of a k-optimal path partition of D. A ***k*-partial coloring** $\mathcal{C}^k$ is a set of k disjoint stable sets called **color classes** (empty color classes are allowed). The **weight** of a k-partial coloring $\mathcal{C}^k$, denoted by $||\mathcal{C}^k||$, is defined as $||\mathcal{C}^k|| = \sum_{C \in \mathcal{C}^k} |C|$. We say that $\mathcal{C}^k$ is an **optimal *k*-partial coloring** if there is no k-partial coloring $\mathcal{B}^k$ such that $||\mathcal{B}^k|| > ||\mathcal{C}^k||$. We denote by $\alpha_k(D)$ the weight of an optimal k-partial coloring of D. Given these definitions, what Greene and Kleitman [Greene and Kleitman 1976] showed was that for every transitive acyclic digraph D, we have $\pi_k(D) = \alpha_k(D)$. Note that $\pi(D) = \pi_1(D)$ and $\alpha(D) = \alpha_1(D)$. Thus, Dilworth's Theorem is a particular case of Greene-Kleitman's Theorem in which $k = 1$.

As Gallai-Milgram's Theorem extends Dilworth's Theorem, it is a natural question whether Greene-Kleitman's Theorem can be extended to digraphs in general. More precisely, is it true that for every digraph D we have that $\pi_k(D) \leq \alpha_k(D)$? Linial [Linial 1981] conjectured that the answer for this question is positive.

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[1]Usually $|P|$ denotes the length of a path (number of arcs), but here it denotes the number of vertices.

Linial's Conjecture [Linial 1981]. *Let D be a digraph and k be a positive integer. Then, $\pi_k(D) \leq \alpha_k(D)$.*

Linial's Conjecture remains open, but we know it holds for acyclic digraphs [Saks 1979], bipartite digraphs [Berge 1982], digraphs which contain a Hamiltonian path [Berge 1982], $k = 1$ [Linial 1978], $k = 2$ [Berger and Hartman 2008] and $k \geq \lambda(D) - 3$ [Herskovics 2013]. In this paper we give partial results on Linial's Conjecture for split digraphs.

2. Split digraphs

Let D be a digraph and let $X \subseteq V(D)$. We denote by $D[X]$ the subdigraph of D induced by X. A digraph D is a **split digraph** if there is a partition of $V(D)$ into two sets X and Y, such that $D[X]$ is a tournament and $D[Y]$ is a stable set. We shall use the notation $D[X, Y]$ to indicate that D is a split digraph with such partition $\{X, Y\}$.

In this section we shall prove an approximation to Linial's Conjecture for split digraphs, i. e., that $\pi_k(D) \leq \alpha_k(D) + 1$ for every split digraph D, as stated in Theorem 1. For that, we need Rédei's Theorem and Lemmas 1 and 2 below.

Rédei's Theorem [Rédei 1934]. *Every tournament contains a Hamiltonian path.*

Lemma 1. *Let $D[X, Y]$ be a split digraph. Then, $\pi_k(D) \leq |Y| + \min\{|X|, k\}$.*
Proof. By Rédei's Theorem, the tournament $D[X]$ contains a path P such that $V(P) = X$. Let $\mathcal{P} = \{P\} \cup \{(y) : y \in Y\}$. Clearly, $\mathcal{P}$ is a path partition of D for which $|\mathcal{P}|_k = \min\{|X|, k\} + |Y|$. Therefore, $\pi_k(D) \leq |\mathcal{P}|_k = \min\{|X|, k\} + |Y|$. ■

Lemma 2. *Let $D[X, Y]$ be a split digraph. Then, $\alpha_k(D) \geq |Y| + \min\{|X|, k - 1\}$. Moreover, when $|X| < k$, we have that $\alpha_k(D) = |V(D)|$.*
Proof. First, suppose that $|X| \leq k - 1$. Let $\mathcal{C}^k = \{Y\} \cup \{\{x\} : x \in X\}$. Note that $\mathcal{C}^k$ is a k-partial coloring of D with $||\mathcal{C}^k|| = |V(D)|$. Therefore, $\alpha_k(D) = ||\mathcal{C}^k|| = |Y| + |X| = |Y| + \min\{|X|, k - 1\}$ and the result follows. We may assume that $|X| \geq k$. Let $\mathcal{C}^k = \{Y\} \cup \{\{x\} : x \in S\}$, where $S \subseteq X$ such that $|S| = k - 1$. Clearly, $\mathcal{C}^k$ is a k-partial coloring for which $||\mathcal{C}^k|| = |Y| + k - 1$. Therefore, $\alpha_k(D) \geq ||\mathcal{C}^k|| = |Y| + k - 1 = |Y| + \min\{|X|, k - 1\}$. ■

Theorem 1. *Let $D[X, Y]$ be a split digraph. Then, $\pi_k(D) \leq \alpha_k(D) + 1$.*
Proof. The result follows immediately from Lemmas 1 and 2. ■

In Section 2.1 we introduce k-loose digraphs and show that Linial's Conjecture holds for them and in Section 2.2 we show that it holds for spider digraphs [Hoàng 1985].

2.1. *k*-loose digraphs

A split digraph $D[X, Y]$ is ***k*-loose** if either $|X| < k$ or there is a $S \subseteq X$ such that $|S| = k$ and no vertex $y \in Y$ is adjacent to every vertex in S. A split digraph $D[X, Y]$ that is not k-loose is called ***k*-tight**. We show in this section that Linial's Conjecture holds for every k-loose digraph (Theorem 2) and for split digraphs such that $|X| \leq k$ (Theorem 3). For that, we need Lemmas 3 and 4 below.

Lemma 3. *Let $D[X, Y]$ be a split digraph. Then, D is k-loose if and only if $\alpha_k(D) \geq |Y| + \min\{|X|, k\}$.*

Proof. Consider that D is k-loose. If $|X| < k$, then by Lemma 2, $\alpha_k(D) = |V(D)| = |Y| + |X| \geq |Y| + \min\{|X|, k\}$. We may thus assume that $|X| \geq k$ and there is $S \subseteq X$ such that $|S| = k$ and no vertex $y \in Y$ is adjacent to every vertex in S. Assume $S = \{x_1, x_2, \ldots, x_k\}$ and let $\mathcal{C}_0^k = \{C_1, C_2, \ldots, C_k\}$ be a k-partial coloring where $C_i = \{x_i\}$ for $i = 1, 2, \ldots, k$. For each $y \in Y$ choose some vertex x_i not adjacent to y (which exists by definition) and add y in color class C_i. The k-partial coloring $\mathcal{C}^k$ thus obtained has weight $|Y| + k = |Y| + \min\{|X|, k\}$ as expected.

Conversely, consider that $\alpha_k(D) \geq |Y| + \min\{|X|, k\}$. If $|X| < k$, then D is k-loose by definition. So, we may assume that $|X| \geq k$ and, whence, $\alpha_k(D) \geq |Y| + k$. We conclude that $\mathcal{C}^k$ must have exactly k vertices of X, besides all $|Y|$ vertices from Y. Let $S = \{x : x \in C_i \cap X \text{ for } i = 1, 2, \ldots, k\}$. Since all vertices of Y belong to $\mathcal{C}^k$, then there is no vertex in Y which is adjacent to every vertex of S. Therefore, D is k-loose. ■

Theorem 2. *Let $D[X, Y]$ be a k-loose split digraph. Then, $\pi_k(D) \leq \alpha_k(D)$.*
Proof. By Lemma 3, $\alpha_k(D) \geq |Y| + \min\{|X|, k\}$. On the other hand, by Lemma 1 $\pi_k(D) \leq |Y| + \min\{|X|, k\}$ and the result follows. ■

Lemma 4. *Let $D[X, Y]$ be a split digraph such that $\lambda(D) > |X|$. Then, $\pi_k(D) \leq \alpha_k(D)$.*
Proof. If $\alpha_k(D) = |V(D)|$, then the result follows trivially. Thus, we may assume that $\alpha_k(D) < |V(D)|$. By Lemma 2 we have that $|X| \geq k$ and also that $\alpha_k(D) \geq |Y| + \min\{|X|, k-1\} = |Y| + k - 1$. Since $\lambda(D) > |X|$, there exists a path P in D such that $|P| = |X| + 1$. Let $\mathcal{P} = \{P\} \cup \{(v) : v \notin V(P)\}$. Clearly, $\mathcal{P}$ is a path partition of D and $|\mathcal{P}|_k = |Y| + k - 1$. Therefore, $\pi_k(D) \leq |\mathcal{P}|_k \leq \alpha_k(D)$. ■

Theorem 3. *Let $D[X, Y]$ be a split digraph such that $|X| \leq k$. Then, $\pi_k(D) \leq \alpha_k(D)$.*
Proof. If D is k-loose, then the result follows by Theorem 2. So, we may assume that D is not k-loose. Hence, $|X| = k$ and there exists a vertex $y \in Y$ which is adjacent to every vertex of X. Therefore, $D[X \cup \{y\}]$ is a tournament and by Rédei's Theorem it has a Hamiltonian path P such that $|P| = |X| + 1$. As P is a path in D as well, we conclude that $\lambda(D) \geq |X| + 1$ and the result follows by Lemma 4. ■

2.2. Spider digraphs

We denote by $\mathcal{N}(v)$ the set of vertices that are adjacent to $v \in V(D)$ (regardless the direction of the arcs). A split digraph $D[X, Y]$ is **spider** [Hoàng 1985] if (i) $|X| = |Y| \geq 2$; and (ii) there exists a bijective function $f : X \to Y$ such that either $\mathcal{N}(x) = \{f(x)\}$ for all $x \in X$ (in this case, we say that D is a **thin** spider) or $\mathcal{N}(x) = Y - f(x)$ for all $x \in X$ (in this case, we say that D is a **thick** spider). Note that thin spider digraphs are k-loose, but thick spider digraphs are k-tight, as long as $|X| > k$. The following theorem shows that Linial's Conjecture holds for spider digraphs.

Theorem 4. *Let $D[X, Y]$ be a spider digraph. Then, $\pi_k(D) \leq \alpha_k(D)$.*
Proof. Let $\ell = |X| = |Y|$. If $\ell \leq k$, then the result follows by Theorem 3. Thus, we may assume that $|X| > k$. Clearly, $\pi_k(D) \leq |V(D)|$ and we deduce that $\alpha_k < |V(D)|$. If D is a thin spider digraph, whence k-loose, the result follows by Theorem 2. Therefore, we may assume that D is a thick spider graph. Since $D[X]$ is a tournament, by Rédei's Theorem, there exists a path P such that $V(P) = X$. Let $P = (x_1, x_2, \ldots, x_\ell)$. Since D is a thick spider digraph, there exists one single vertex $y_i \in Y$ that is not adjacent to x_i, for $i = 1, \ldots, \ell$. Note that if $\lambda(D) > |X|$, then the result follows by Lemma 4. So we may assume that $\lambda(D) \leq |X|$.

Let Px_i denote the subpath $(x_1, x_2, \ldots, x_i)$ and let x_iP denote the subpath $(x_i, x_{i+1}, \ldots, x_\ell)$. We denote by $W \circ Q$ the concatenation of two paths W and Q.

Claim 1: *If $x_i \in X$, $y_j \in Y$ and $i < j$, then $(x_i, y_j) \in A(D)$.*
We prove this claim by induction on i. If $i = 1$, assume by contradiction that $(y_j, x_1) \in A(D)$; then $P' = (y_j, x_1) \circ P$ is a path in D such that $|P'| = |X| + 1$, a contradiction. Hence, $(x_1, y_j) \in A(D)$. Consider now $i > 1$. Recall that y_j is adjacent to every vertex in $X - \{x_j\}$. Thus, y_j is adjacent to every vertex of $V(Px_i)$. By induction hypothesis, we have $(x_{i-1}, y_j) \in A(D)$. Suppose by contradiction that $(y_j, x_i) \in A(D)$. Then, there is a path $P' = Px_{i-1} \circ (x_{i-1}, y_j, x_i) \circ x_iP$ such that $|P'| = |X| + 1$, a contradiction. Therefore, $(x_i, y_j) \in A(D)$. This completes the proof of Claim 1.

Claim 2: *If $x_i \in X$, $y_j \in Y$ and $j < i$, then $(y_j, x_i) \in A(D)$.*
We omit the proof of Claim 2, as it is analogous to that of Claim 1.

We claim that both $P_0 = (x_1, y_2, x_3, y_4, \ldots)$ and $P_1 = (y_1, x_2, y_3, x_4, \ldots)$ are paths in D. By Claim 1 we have that $(x_i, y_{i+1}) \in A(D)$ for $i = 1, 3, \ldots$, and by Claim 2 we have that $(y_j, x_{j+1}) \in A(D)$ for $j = 2, 4, \ldots$. Hence P_0 is a path in D. The proof is analogous for P_1. Clearly, $\mathcal{P} = \{P_0, P_1\}$ is a path partition of D. Moreover, $|P_0| = |P_1| = \ell$ and $|\mathcal{P}|_k = 2\min\{\ell, k\} = 2k$. Since $|X| > k$, we have that $\min\{\ell, k\} = k \leq |X| - 1 = |Y| - 1$. Thus, $|\mathcal{P}|_k = 2k \leq k + |Y| - 1$. On the other hand, by Lemma 2, $\alpha_k(D) \geq |Y| + \min\{|X|, k - 1\} = |Y| + k - 1$. Therefore, $\pi_k(D) \leq |\mathcal{P}|_k = 2k \leq |Y| + k - 1 \leq \alpha_k(D)$. ■

3. Conclusion

We showed that Linial's Conjecture holds for k-loose digraphs and for some subclasses of k-tight digraphs, namely those with $|X| = k$ and the thick spider digraphs. It is easy to see that for k-tight digraphs, $\alpha_k(D) = |Y| + k - 1$. Therefore, it is clear that any approach to prove Linial's Conjecture for k-tight digraphs must involve finding a path partition with k-norm less than or equal to $|Y| + k - 1$. We are currently working on this idea.

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