Some families of 0-rotatable graceful caterpillars

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Abstract. *A graceful labelling of a tree T is an injective function $f\colon V(T) \to \{0, 1, \ldots, |E(T)|\}$ such that $\{|f(u) - f(v)|\colon uv \in E(T)\} = \{1, 2, \ldots, |E(T)|\}$. A tree T is said to be 0-rotatable if, for any $v \in V(T)$, there exists a graceful labelling f of T such that $f(v) = 0$. In this work, it is proved that the following families of caterpillars are 0-rotatable: caterpillars with perfect matching; caterpillars obtained by identifying a central vertex of a path P_n with a vertex of K_2; caterpillars obtained by identifying one leaf of the star $K_{1,s-1}$ to a leaf of P_n, with $n \geq 4$ and $s \geq \lceil \frac{n-1}{2} \rceil$; caterpillars with diameter five or six; and some families of caterpillars with diameter at least seven. This result reinforces the conjecture that all caterpillars with diameter at least five are 0-rotatable.*

1. Introduction

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A graceful labelling of G is an injection $f\colon V(G) \to \{0, 1, \ldots, |E(G)|\}$ such that $\{|f(u) - f(v)|\colon uv \in E(G)\} = \{1, 2, \ldots, |E(G)|\}$. We say that G is *graceful* if it has a graceful labelling.

In 1967, Rosa introduced four types of labellings of graphs, including graceful labellings, and posed the *Graceful Tree Conjecture* which states that all trees are graceful [Rosa 1967]. Rosa proved that the Graceful Tree Conjecture is a strenghtened version of the well-known *Ringel-Kotzig Conjecture* which states that the complete graph K_{2m+1} has a cyclic decomposition into subgraphs isomorphic to a given tree T with m edges. The Graceful Tree Conjecture is a very important open problem in Graph Theory, with more than a thousand papers about it [Gallian 2015].

As soon as one starts investigating graceful labellings of trees, it becomes clear the importance of knowing how to construct graceful labellings with the label 0 appearing in a given vertex. The importance of label 0 in a graceful labelling of a tree T is due to the fact that it is easy to grow T by adding k new leaves to the 0-labelled vertex and expand the graceful labelling by assigning labels $|E(T)| + 1, \ldots, |E(T)| + k$ to these new leaves. A tree T is 0-*rotatable* if, for any $v \in V(T)$, there exists a graceful labelling f of T such that $f(v) = 0$.

The importance of 0-rotatability of trees was first noted by Rosa in his seminal paper [Rosa 1967], in which the author stated, without proof, that all paths are 0-rotatable. Ten years later, the author published a proof of this result [Rosa 1977]. Meanwhile, in 1969, some examples of non 0-rotatable trees were discovered [Gallian 2015]. As an example, the smallest non 0-rotatable tree is the tree obtained by identifying one leaf of the star $K_{1,3}$ to a leaf of P_3. Posteriorly, Chung and Hwang investigated the 0-rotatability of a product of trees called Δ-*construction* and proved that if two trees T_1 and T_2 are 0-rotatable, then their product $T_1 \Delta T_2$ is also 0-rotatable [Chung and Hwang 1981]. In 2004, Bussel [Bussel 2004] showed that all trees with diameter at most three are 0-rotatable. The author also showed that there exist non 0-rotatable trees with diameter four. In fact, he completely characterized the diamater-four non 0-rotatable trees using the following result.

Theorem 1 ([Bussel 2004]). *Let T be a tree of diameter four such that its center v has degree two. Let v_1, v_2 be the vertices adjacent to v and m_1, m_2 be the number of leaves adjacent to v_1, v_2, respectively. Assume $m_1 \geq m_2$. The tree T has a graceful labelling f with $f(v) = 0$ if and only if there exist integers x and r such that $m_1 = (m_2 + 2 - x)(r - 1) - x$, with: (i) x, r not both odd; (ii) $2 \leq r \leq |E(T)|/2$; and (iii) $0 \leq x \leq \min\{r - 1, m_2\}$.* □

Let $\mathcal{D}$ denote the class of diameter-four trees whose center has degree two and that do not satisfy the conditions of Theorem 1. Let $\mathcal{D}'$ be the class of trees built by identifying a leaf of an arbitrary path P_n, $n \geq 1$, with the center of a tree in $\mathcal{D}$. Bussel proved that, given a tree T with diameter four, T is 0-rotatable if and only if $T \notin \mathcal{D}'$. Additionally, he showed that all trees with at most 14 vertices and that are not 0-rotatable belong to the class $\mathcal{D}'$. Thus, based on these results, the author posed the following conjecture.

Conjecture 2 ([Bussel 2004]). *The class $\mathcal{D}'$ contains all non 0-rotatable trees.*

From the time it was first studied, 50 years ago, 0-rotatability of trees has been considered a possible way to approach the Graceful Tree Conjecture, and also a challenging problem by itself. In particular, a family of trees for which the 0-rotatability property is not known is the family of caterpillars, defined as follows. A tree T is a *caterpillar* if either T is a path or the subgraph obtained by deleting all its leaves (the *base* of T) is a path.

In fact, note that, if Conjecture 2 is true, then it implies that every caterpillar with diamater at least five is 0-rotatable. Considering these observations, in this work, we investigate Conjecture 2 restricted to caterpillars and prove that the following families of caterpillars are 0-rotatable: (i) caterpillars with perfect matching; (ii) caterpillars obtained by identifying a central vertex of P_n with a vertex of K_2; (iii) caterpillars obtained by identifying one leaf of $K_{1,s-1}$ to one leaf of P_n, with $n \geq 4$ and $s \geq \lceil \frac{n-1}{2} \rceil$; (iv) caterpillars with diameter five or six; and (v) some families of caterpillars with diameter at least seven. These results reinforce Conjecture 2.

2. Preliminaries

A *matching* of a graph G is a set of pairwise nonadjacent edges of G. Let M be a matching of a graph G. A vertex $v \in V(G)$ is *saturated* by M if v is incident with an edge of M. A *perfect matching* of G is a matching that saturates all the vertices of G. Let T be a tree with a perfect matching M. The *contree* of T is the tree T' obtained from T by contracting all the edges of M.

Broersma and Hoede [Broersma and Hoede 1999] introduced the concept of strongly graceful labellings of trees defined as follows. Let T be a tree with a perfect matching M. A labelling f of T is *strongly graceful* if f is a graceful labelling and if $f(u)+f(v) = |E(T)|$ for every edge $uv \in M$. The authors proved that the Graceful Tree Conjecture is true if and only if every tree with a perfect matching has a strongly graceful labelling. They also studied the label 0 in strong graceful labellings, as presented in the next lemma. This result is important for the proof of Theorem 5.

Lemma 3 ([Broersma and Hoede 1999])**.** *Let T be a tree with a perfect matching M and $uv \in M$, $u, v \in V(T)$. Let T' be the contree of T and let $x \in V(T')$ be the vertex corresponding to the edge uv. If T' has a graceful labelling f', with $f'(x) = 0$, then T has two strongly graceful labellings f_1 and f_2, such that: (i) $f_1(u) = 0$ and $f_1(v) = |E(T)|$; (ii) $f_2(u) = |E(T)|$ and $f_2(v) = 0$.* □

Given a graceful labelling f of a tree T, the *complementary labelling* of f is the labelling $\overline{f}$ defined by $\overline{f}(v) = |E(T)| - f(v)$ for each $v \in V(T)$. Note that the complementary labelling is also a graceful labelling since: (i) $f(v)$ is an injection from $V(T)$ to $\{0, \ldots, |E(T)|\}$; and (ii) for each $uv \in E(T)$, $|\overline{f}(u) - \overline{f}(v)| = |(m - f(u)) - (m - f(v))| = |f(v) - f(u)|$.

A technique used in our proofs is the method of transfers, defined as follows. Let u, v, u_1 be distinct vertices of a tree T, such that u_1 is adjacent to u. We call *transfer*, the operation of deleting the edge u_1u from T and adding the edge u_1v. After the transfer operation, we say that u_1 has been *moved* from u to v. The following lemma determines when a transfer performed over a graceful tree generates another graceful tree.

Lemma 4 ([Hrnčiar and Haviar 2001])**.** *Let f be a graceful labelling of a tree T and let $u, v \in V(T)$ be two distinct vertices. If u is adjacent to leaves $u_1, u_2 \in V(T)$, such that $u_1 \neq v$, $u_2 \neq v$ and $f(u_1) + f(u_2) = f(u) + f(v)$, then the tree T' obtained by moving u_1, u_2 from u to v is also graceful.* □

3. Results

In this section, we state our main results. In particular, Theorems 7 and 8, and the second family stated in Theorem 6 show that, for each integer $d \geq 5$, there exist 0-rotatable caterpillars with diameter d and an arbitrarily large number of vertices. These results reinforce the conjecture that all caterpillars with diameter at least five are 0-rotatable.

Theorem 5. *Every caterpillar with a perfect matching is 0-rotatable.*

Proof. Let T be a caterpillar with a perfect matching M and let $uv \in M$, $u, v \in V(T)$. Let T' be the contree of T and let $x \in V(T')$ be the vertex corresponding to the edge uv. Since T has a perfect matching, we have that T' is a path. Rosa proved that every path is 0-rotatable [Rosa 1977]. Therefore, T' is 0-rotatable. Hence, T' has a graceful labelling f' such that $f'(x) = 0$. By Lemma 3, T has two strongly graceful labellings f_1 and f_2 such that: $f_1(u) = 0$ and $f_1(v) = |E(T)|$; $f_2(u) = |E(T)|$ and $f_2(v) = 0$. Therefore, there exist strongly graceful labellings of T which assign the label 0 to vertex u or v. Since uv is arbitrary, we conclude that T is 0-rotatable. □

Theorem 6. *The following families of caterpillars are 0-rotatable: (i) caterpillars obtained by identifying a vertex of K_2 with a central vertex of P_n; (ii) caterpillars obtained by identifying one leaf of the star $K_{1,s-1}$ to a leaf of P_n, with $n \geq 4$ and $s \geq \lceil \frac{n-1}{2} \rceil$.*

Outline of the proof. Let T be a caterpillar as defined in the hypothesis and let $v \in V(T)$ be an arbitrary vertex. First, we specify an edge $wz \in E(T)$, $w, z \in V(T)$, and remove

wz from T, thus obtaining two vertex-disjoint subgraphs $H_1 \subset T$ and $H_2 \subset T$ such that $w, v \in V(H_1)$ and $z \in V(H_2)$. Consider a bipartition $\{V_1, V_2\}$ of $V(H_1)$ such that $v \in V_1$ and define $k = |V_1|$. Thus, we construct injective labellings f_1, f_2 for H_1, H_2, respectively, where $f_1\colon V(H_1) \to \{0, \ldots, k-1\} \cup \{k + |E(H_2)| + 1, \ldots, |E(T)|\}$, $f_2\colon V(H_2) \to \{k, k+1, \ldots, k+|E(H_2)|\}$, and such that: (i) $f_1(v) = 0$; (ii) the edge labels induced by f_2 are $1, 2, \ldots, |E(H_2)|$; (iii) the edge labels induced by f_1 are $|E(H_2)| + 2, \ldots, |E(T)|$; and (iv) $f_1(w)$ and $f_2(z)$ are such that $|f_1(w) - f_2(z)| = |E(H_2)| + 1$. Finally, we define a labelling f of T as follows: for $u \in V(T)$, $f(u) = f_1(u)$ if $u \in H_1$; and $f(u) = f_2(u)$ if $u \in H_2$. Therefore, f is a graceful labelling of T with $f(v) = 0$ and, since v is an arbitrary vertex, we obtain that T is 0-rotatable. □

Theorem 7. *If T is a caterpillar with diameter five or six, then T is 0-rotatable.*

Outline of the proof. Let T be a caterpillar with diameter five or six. For each vertex $v \in V(T)$ in the base of T, we construct a graceful labelling f of T that assigns label 0 to v and assigns label $|E(T)|$ to any leaf $u \in V(T)$ adjacent to v. Consequently, given any of these graceful labellings f, one can use its complementary labelling $\overline{f}$ in order to obtain $\overline{f}(u) = 0$ and $\overline{f}(v) = |E(T)|$. Since $\overline{f}$ is also a graceful labelling and f was constructed considering an arbitrary vertex v of the base of T, we obtain that T is 0-rotatable. □

Theorem 8. *If T has odd diameter at least seven and each vertex of its base is adjacent to a positive even number of leaves, then T is 0-rotatable. Additionally, If T has even diameter at least eight and each vertex of its base is adjacent to an even number of at least 4 leaves, then T is 0-rotatable.*

Outline of the proof. The technique used in this proof is similar to the technique used in the proof of Theorem 7. □

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