Some families of 0-rotatable graceful caterpillars

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Abstract. A graceful labelling of a tree $T$ is an injective function $f: V(T) \to \{0, 1, \ldots, |E(T)|\}$ such that $\{|f(u) - f(v)|: uv \in E(T)\} = \{1, 2, \ldots, |E(T)|\}$. A tree $T$ is said to be 0-rotatable if, for any $v \in V(T)$, there exists a graceful labelling $f$ of $T$ such that $f(v) = 0$. In this work, it is proved that the following families of caterpillars are 0-rotatable: caterpillars with perfect matching; caterpillars obtained by identifying a central vertex of a path $P_n$ with a vertex of $K_2$; caterpillars obtained by identifying one leaf of the star $K_{1,s-1}$ to a leaf of $P_n$, with $n \geq 4$ and $s \geq \lceil \frac{n-1}{2} \rceil$; caterpillars with diameter five or six; and some families of caterpillars with diameter at least seven. This result reinforces the conjecture that all caterpillars with diameter at least five are 0-rotatable.

1. Introduction

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A graceful labelling of $G$ is an injection $f: V(G) \to \{0, 1, \ldots, |E(G)|\}$ such that $\{|f(u) - f(v)|: uv \in E(G)\} = \{1, 2, \ldots, |E(G)|\}$. We say that $G$ is graceful if it has a graceful labelling.

In 1967, Rosa introduced four types of labellings of graphs, including graceful labellings, and posed the Graceful Tree Conjecture which states that all trees are graceful [Rosa 1967]. Rosa proved that the Graceful Tree Conjecture is a strengthened version of the well-known Ringel-Kotzig Conjecture which states that the complete graph $K_{2m+1}$ has a cyclic decomposition into subgraphs isomorphic to a given tree $T$ with $m$ edges. The Graceful Tree Conjecture is a very important open problem in Graph Theory, with more than a thousand papers about it [Gallian 2015].

As soon as one starts investigating graceful labellings of trees, it becomes clear the importance of knowing how to construct graceful labellings with the label 0 appearing in a given vertex. The importance of label 0 in a graceful labelling of a tree $T$ is due to the fact that it is easy to grow $T$ by adding $k$ new leaves to the 0-labelled vertex and expand the graceful labelling by assigning labels $|E(T)| + 1, \ldots, |E(T)| + k$ to these new leaves. A tree $T$ is 0-rotatable if, for any $v \in V(T)$, there exists a graceful labelling $f$ of $T$ such that $f(v) = 0$.

The importance of 0-rotatability of trees was first noted by Rosa in his seminal paper [Rosa 1967], in which the author stated, without proof, that all paths are 0-rotatable. Ten years later, the author published a proof of this result [Rosa 1977]. Meanwhile, in 1969, some examples of non 0-rotatable trees were discovered [Gallian 2015]. As an
example, the smallest non 0-rotatable tree is the tree obtained by identifying one leaf of the star $K_{1,3}$ to a leaf of $P_5$. Posterily, Chung and Hwang investigated the 0-rotatability of a product of trees called $\Delta$-construction and proved that if two trees $T_1$ and $T_2$ are 0-rotatable, then their product $T_1 \Delta T_2$ is also 0-rotatable [Chung and Hwang 1981]. In 2004, Bussel [Bussel 2004] showed that all trees with diameter at most three are 0-rotatable. The author also showed that there exist non 0-rotatable trees with diameter four. In fact, he completely characterized the diamater-four non 0-rotatable trees using the following result.

**Theorem 1** ([Bussel 2004]). *Let $T$ be a tree of diameter four such that its center $v$ has degree two. Let $v_1, v_2$ be the vertices adjacent to $v$ and $m_1, m_2$ be the number of leaves adjacent to $v_1, v_2$, respectively. Assume $m_1 \geq m_2$. The tree $T$ has a graceful labelling $f$ with $f(v) = 0$ if and only if there exist integers $x$ and $r$ such that $m_1 = (m_2 + 2 - x)(r - 1) - x$, with: (i) $x, r$ not both odd; (ii) $2 \leq r \leq |E(T)|/2$; and (iii) $0 \leq x \leq \min\{r - 1, m_2\}$.\qed*

Let $D$ denote the class of diameter-four trees whose center has degree two and that do not satisfy the conditions of Theorem 1. Let $D'$ be the class of trees built by identifying a leaf of an arbitrary path $P_n$, $n \geq 1$, with the center of a tree in $D$. Bussel proved that, given a tree $T$ with diameter four, $T$ is 0-rotatable if and only if $T \not\in D'$. Additionally, he showed that all trees with at most 14 vertices and that are not 0-rotatable belong to the class $D'$. Thus, based on these results, the author posed the following conjecture.

**Conjecture 2** ([Bussel 2004]). *The class $D'$ contains all non 0-rotatable trees.*

From the time it was first studied, 50 years ago, 0-rotatability of trees has been considered a possible way to approach the Graceful Tree Conjecture, and also a challenging problem by itself. In particular, a family of trees for which the 0-rotatability property is not known is the family of caterpillars, defined as follows. A tree $T$ is a *caterpillar* if either $T$ is a path or the subgraph obtained by deleting all its leaves (the *base* of $T$) is a path.

In fact, note that, if Conjecture 2 is true, then it implies that every caterpillar with diameter at least five is 0-rotatable. Considering these observations, in this work, we investigate Conjecture 2 restricted to caterpillars and prove that the following families of caterpillars are 0-rotatable: (i) caterpillars with perfect matching; (ii) caterpillars obtained by identifying a central vertex of $P_n$ with a vertex of $K_2$; (iii) caterpillars obtained by identifying one leaf of $K_{1,s-1}$ to one leaf of $P_n$, with $n \geq 4$ and $s \geq \lceil n/2 \rceil$; (iv) caterpillars with diameter five or six; and (v) some families of caterpillars with diameter at least seven. These results reinforce Conjecture 2.

2. Preliminaries

A *matching* of a graph $G$ is a set of pairwise nonadjacent edges of $G$. Let $M$ be a matching of a graph $G$. A vertex $v \in V(G)$ is *saturated* by $M$ if $v$ is incident with an edge of $M$. A *perfect matching* of $G$ is a matching that saturates all the vertices of $G$. Let $T$ be a tree with a perfect matching $M$. The *contre of $T$* is the tree $T'$ obtained from $T$ by contracting all the edges of $M$.

Broersma and Hoede [Broersma and Hoede 1999] introduced the concept of strongly graceful labellings of trees defined as follows. Let $T$ be a tree with a perfect
matching $M$. A labelling $f$ of $T$ is strongly graceful if $f$ is a graceful labelling and if $f(u) + f(v) = |E(T)|$ for every edge $uv \in M$. The authors proved that the Graceful Tree Conjecture is true if and only if every tree with a perfect matching has a strongly graceful labelling. They also studied the label 0 in strong graceful labellings, as presented in the next lemma. This result is important for the proof of Theorem 5.

**Lemma 3** ([Broersma and Hoede 1999]). Let $T$ be a tree with a perfect matching $M$ and $uv \in M$, $u,v \in V(T)$. Let $T'$ be the contree of $T$ and let $x \in V(T')$ be the vertex corresponding to the edge $uv$. If $T'$ has a graceful labelling $f'$, with $f'(x) = 0$, then $T$ has two strongly graceful labellings $f_1$ and $f_2$, such that: (i) $f_1(u) = 0$ and $f_1(v) = |E(T)|$; (ii) $f_2(u) = |E(T)|$ and $f_2(v) = 0$.

Given a graceful labelling $f$ of a tree $T$, the complementary labelling of $f$ is the labelling $\overline{f}$ defined by $\overline{f}(v) = |E(T)| - f(v)$ for each $v \in V(T)$. Note that the complementary labelling is also a graceful labelling since: (i) $f(v)$ is an injection from $V(T)$ to $\{0, \ldots, |E(T)|\}$; and (ii) for each $uv \in E(T)$, $|\overline{f}(u) - \overline{f}(v)| = |(m - f(u)) - (m - f(v))| = |f(v) - f(u)|$.

A technique used in our proofs is the method of transfers, defined as follows. Let $u,v,u_1$ be distinct vertices of a tree $T$, such that $u_1$ is adjacent to $u$. We call transfer, the operation of deleting the edge $u_1u$ from $T$ and adding the edge $u_1v$. After the transfer operation, we say that $u_1$ has been moved from $u$ to $v$. The following lemma determines when a transfer performed over a graceful tree generates another graceful tree.

**Lemma 4** ([Hrnčiar and Haviar 2001]). Let $f$ be a graceful labelling of a tree $T$ and let $u,v \in V(T)$ be two distinct vertices. If $u$ is adjacent to leaves $u_1,u_2 \in V(T)$, such that $u_1 \neq v,u_2 \neq v$ and $f(u_1) + f(u_2) = f(u) + f(v)$, then the tree $T'$ obtained by moving $u_1,u_2$ from $u$ to $v$ is also graceful.

### 3. Results

In this section, we state our main results. In particular, Theorems 7 and 8, and the second family stated in Theorem 6 show that, for each integer $d \geq 5$, there exist $0$-rotatable caterpillars with diameter $d$ and an arbitrarily large number of vertices. These results reinforce the conjecture that all caterpillars with diameter at least five are $0$-rotatable.

**Theorem 5.** Every caterpillar with a perfect matching is $0$-rotatable.

**Proof.** Let $T$ be a caterpillar with a perfect matching $M$ and let $uv \in M$, $u,v \in V(T)$. Let $T'$ be the contree of $T$ and let $x \in V(T')$ be the vertex corresponding to the edge $uv$. Since $T$ has a perfect matching, we have that $T'$ is a path. Rosa proved that every path is $0$-rotatable [Rosa 1977]. Therefore, $T'$ is $0$-rotatable. Hence, $T'$ has a graceful labelling $f'$ such that $f'(x) = 0$. By Lemma 3, $T$ has two strongly graceful labellings $f_1$ and $f_2$ such that: $f_1(u) = 0$ and $f_1(v) = |E(T)|$; $f_2(u) = |E(T)|$ and $f_2(v) = 0$. Therefore, there exist strongly graceful labellings of $T$ which assign the label 0 to vertex $u$ or $v$. Since $uv$ is arbitrary, we conclude that $T$ is $0$-rotatable.

**Theorem 6.** The following families of caterpillars are $0$-rotatable: (i) caterpillars obtained by identifying a vertex of $K_2$ with a central vertex of $P_n$; (ii) caterpillars obtained by identifying one leaf of the star $K_{1,s-1}$ to a leaf of $P_n$ with $n \geq 4$ and $s \geq \lceil \frac{n-1}{2} \rceil$.

**Outline of the proof.** Let $T$ be a caterpillar as defined in the hypothesis and let $v \in V(T)$ be an arbitrary vertex. First, we specify an edge $wz \in E(T)$, $w,z \in V(T)$, and remove
when a transfer performed over a graceful tree generates another graceful tree. Consider a bipartition \( \{V_1, V_2\} \) of \( V(H_1) \) such that \( v \in V_1 \) and define \( k = |V_1| \). Thus, we construct injective labellings \( f_1, f_2 \) for \( H_1, H_2 \), respectively, where \( f_1: V(H_1) \rightarrow \{0, \ldots, k - 1\} \cup \{k + |E(H_2)| + 1, \ldots, |E(T)|\} \), \( f_2: V(H_2) \rightarrow \{k, k + 1, \ldots, k + |E(H_2)|\} \), and such that: (i) \( f_1(v) = 0 \); (ii) the edge labels induced by \( f_2 \) are 1, 2, \ldots, \( |E(H_2)| \); (iii) the edge labels induced by \( f_1 \) are \( |E(H_2)| + 2, \ldots, |E(T)| \); and (iv) \( f_1(w) \) and \( f_2(z) \) are such that \( |f_1(w) - f_2(z)| = |E(H_2)| + 1 \). Finally, we define a labelling \( f \) of \( T \) as follows: for \( u \in V(T) \), \( f(u) = f_1(u) \) if \( u \in H_1 \); and \( f(u) = f_2(u) \) if \( u \in H_2 \). Therefore, \( f \) is a graceful labelling of \( T \) with \( f(v) = 0 \) and, since \( v \) is an arbitrary vertex, we obtain that \( T \) is 0-rotatable.

**Theorem 7.** If \( T \) is a caterpillar with diameter five or six, then \( T \) is 0-rotatable.

**Outline of the proof.** Let \( T \) be a caterpillar with diameter five or six. For each vertex \( v \in V(T) \) in the base of \( T \), we construct a graceful labelling \( f \) of \( T \) that assigns label 0 to \( v \) and assigns label \( |E(T)| \) to any leaf \( u \in V(T) \) adjacent to \( v \). Consequently, given any of these graceful labellings \( f \), one can use its complementary labelling \( \overline{f} \) in order to obtain \( \overline{f}(u) = 0 \) and \( \overline{f}(v) = |E(T)| \). Since \( \overline{f} \) is also a graceful labelling and \( f \) was constructed considering an arbitrary vertex \( v \) of the base of \( T \), we obtain that \( T \) is 0-rotatable.

**Theorem 8.** If \( T \) has odd diameter at least seven and each vertex of its base is adjacent to a positive even number of leaves, then \( T \) is 0-rotatable. Additionally, if \( T \) has even diameter at least eight and each vertex of its base is adjacent to an even number of at least 4 leaves, then \( T \) is 0-rotatable.

**Outline of the proof.** The technique used in this proof is similar to the technique used in the proof of Theorem 7.

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References


