Results on Circular-Arc Bigraphs
(Extended Abstract)

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Abstract. We present a series of results related to the structural properties of the bipartite graph class known as circular-arc bigraphs. We also propose the definition of a Helly circular-arc bigraph subclass, based on a concept known as bipartite-Helly, along with a few results related to its structural properties.

1. Introduction

A circular-arc graph is a graph that admits an intersection model of arcs on a circle. Arising as a generalization of interval graphs, the class has been extensively studied by many researchers since 1964 [Lin and Szwarcfiter 2009], yielding a plethora of relevant results. Trotter and Moore [Trotter and Moore 1976] presented infinite sets of minimally non circular-arc graphs. Francis et al. [Francis et al. 2014] presented a characterization of circular-arc graphs, along with a certifying recognition algorithm. Furthermore, Hell and Huang [Hell and Huang 2004] characterized two-clique circular arc graphs as the complements of interval bigraphs. Soulignac, in his thesis [Soulignac 2010], characterized different circular-arc subclasses, including the Helly subclass, based on the concept of Helly families [Helly 1923]. A linear-time algorithm for recognition of the class was presented by McConell [McConnell 2003], based on the so-called circular-arc matrices.

The concept of circular-arc bigraphs arises as a bipartite variation of the circular-arc concept, similar to interval bigraphs with relation to interval graphs. We treat circular-arc bigraph representations as bi-circular-arc models \( B = (C, I, E) \), where \( C \) is a circle, and \( I, E \) are families of arcs over \( C \). It is then said that \( B \) represents a bipartite graph \( G = (U, V, E) \) if there is a one-to-one correspondence between \( U \) and \( I \), and one between \( V \) and \( E \), such that, for every \( u \in U \) and every \( v \in V \), \( \{u, v\} \in E \) if and only if the arc corresponding to \( u \) in \( I \) and the one corresponding \( v \) in \( E \) intersect.

Being a rather recent subject, it has yet to be as extensively studied as the other classes cited. Basu et al. [Basu et al. 2013] made a series of characterizations of circular-arc bigraphs, as well as the proper and unit subclasses, based on the so-called biadjacency matrices. Das et al. [Das and Chakraborty 2015] found a forbidden structure characterization of proper circular-arc bigraphs and interval bigraphs.

In this paper, we present a handful of results about the structural properties of circular-arc bigraphs, as well as the Helly subclass, which we introduce. Our study’s main focus were the structural properties of circular-arc bigraphs, based on their graph structures instead of matrix structures. A handful of potentially useful results were found, which we present in the next section.

\textsuperscript{*}Partially supported by Coordenação de Aperfeiçoamento Pessoal de Nível Superior, CAPES
\textsuperscript{†}Partially supported by ANPCyT PICT-2013-2205 and CONICET
2. Results on Circular-Arc Bigraphs

To represent circular arcs on a circle $C$ of length $n$, we represent each point as a real number $0 \leq r < n$, which denotes a clockwise offset through the circle from a fixed zero point in $C$. It is then possible to denote a circular arc by a pair $(r_1, r_2)$ of real numbers, where the arc is traced clockwise from point $r_1$ to point $r_2$, respectively called the counter-clockwise and clockwise endpoints of the arc. Furthermore, in the enunciation of the following results, every index sum is circular, which means that in an enumerated set $\{a_1, ..., a_n\}$, we have $a_{n+1} = a_1$ and $a_{1-n} = a_n$.

**Lemma 1.** Let $B = (U, V, E)$ be a bipartite graph, with $|V| = n$ and $|U| = m$. If there is an ordering $S = (v_1, ..., v_n)$ of $V$ such that, for every $u \in U$, the neighborhood of $u$ is circularly consecutive with relation to $S$, then $B$ is a circular-arc bigraph.

**Proof.** Start with a circle $C$ of length $n + 1$. For each $v_i \in V$, draw an arc $(i, i+1) \in \mathbb{I}$ to represent it. Then, for each $u \in U$ whose neighborhood ranges from $v_i$ to $v_j$, we draw an arc $(i + \frac{1}{2}, j + \frac{1}{2}) \in \mathbb{E}$ to represent it. $(C, \mathbb{I}, \mathbb{E})$ corresponds to $B$. \hfill \Box

Lemma 1 implies that graphs such as generalized crowns and bipartite permutation graphs are all circular-arc bigraphs.

**Definition 1.** Let $a > 0$ and $0 \leq b < a$. The generalized crown graph $S^a_b$ is a bipartite graph with vertex set $\{u_0, ..., u_{a-1}\} \cup \{v_0, ..., v_{a-1}\}$, such that the neighborhood of $v_i$ is $U - \{u_i, ..., u_{i+b}\}$.

**Definition 2.** Let $G = (V, E)$ be a graph. Then $G$ is a permutation graph if there exist two permutations $P_1, P_2$ of its vertex set such that, if the index of $v$ in $P_1$ is less than that of $w$, then $\{v, w\} \in E$ if and only if the index of $v$ in $P_2$ is greater than that of $w$. A bipartite permutation graph is simply a permutation graph that is bipartite.

Let $\mathbb{A}$ be a family of arcs on a circle $C$. An arc $B$ of $\mathbb{A}$ is said to minimally intersect $A \in \mathbb{A}$ if it intersects $A$ without intersecting another arc $A' \in \mathbb{A}$ such that $A' \subset A$. Lemma 2 allows us to describe a type of forbidden structure for the class.

**Lemma 2.** Let $0 < m < n$, and let $\mathbb{A}$ be a family of arcs on a circle $C$, with $|\mathbb{A}| = n$, such that no two arcs have coinciding endpoints. There are at most $n$ subfamilies $\mathbb{A}' \subset \mathbb{A}$ such that $|\mathbb{A}'| = m$ for which it is possible to draw an arc on $C$ that intersects every arc of $\mathbb{A}'$ without intersecting any of $\mathbb{A} - \mathbb{A}'$.

**Proof.** Let $B$ be an arc on $C$ ($B \notin \mathbb{A}$) that intersects exactly $m$ arcs of $\mathbb{A}$. Let $\mathbb{A}'$ be the family of arcs intersected by $B$, and let $S = (A_1, ..., A_m)$ be the order in which the counter-clockwise endpoint of each arc in $\mathbb{A}'$ is first encountered by tracing the circle clockwise from the clockwise endpoint of $B$. Let $A_i$ be the first arc in the order $S$ that is minimally intersected by $B$. We shall call $A_i$ the first minimally intersected arc of $B$. Notice that $\mathbb{A}'$ contains all arcs that contain $A_i$. Let $k > 0$ be the number of such arcs. In particular, notice that every arc in $\{A_1, ..., A_{i-1}\}$ contains $A_i$, otherwise, either they would be minimally intersected themselves, or they’d contain another arc that is minimally intersected and comes before $A_i$ in $S$. The other arcs that $\mathbb{A}'$ contains will be exactly the first $m - k$ arcs that are not contained in (nor contain) $A_i$ and whose counter-clockwise endpoints come immediately after the counter-clockwise endpoint of $A_i$. Therefore, if $A_i$ is the first minimally intersected arc of a new arc $B$ that intersects $m$ arcs, $\mathbb{A}'$ will always be the same. That implies that for each arc $A \in \mathbb{A}$, there is at
most one subfamily of \( m \) arcs which can be intersected by a new arc with \( A \) being the first minimally intersected arc (there might also be none, for instance, if an arc is contained in more than \( m - 1 \) arcs). Therefore, there are at most \( n \) subfamilies of \( m \) arcs that can be exclusively intersected by a new arc. \( \square \)

**Corollary 3.** Let \( B = (U, V, E) \) be a bipartite graph, such that \( |U| = n, |V| = n + 1 \), and there is a \( k \) such that \( |N(v)| = k \) for all \( v \in V \). If the neighborhoods of the vertices of \( V \) are all pairwise distinct, then \( B \) is not a circular-arc bigraph.

Figure 1 contains a handful of forbidden subgraphs we found for the class.

**Figure 1.** List of forbidden subgraphs we found for the circular-arc bigraph class.

### 2.1. Studies on the Helly Subclass

The concept of Helly families [Helly 1923] lends itself to the definition of many extensively studied graph classes [Szwarcfiter 1997, Groshaus and Szwarcfiter 2007, Groshaus and Szwarcfiter 2008]. An adaptation of the definition of Helly families, known as bipartite-Helly, was formulated by Groshaus and Szwarcfiter [Groshaus and Szwarcfiter 2010], and allows us to define Helly-like properties between pairs of families. For the results in this section, we first introduce interval bigraphs in Definition 3, and then *Helly circular-arc bigraphs* and *Helly interval bigraphs* in Definition 4, loosely based on the bipartite-Helly concept.

**Definition 3.** A bipartite graph \( B = (U, V, E) \) is an interval bigraph if it admits a bi-interval representation. A bi-interval representation of \( B \) is a mapping between \( U \cup V \) and a family of intervals on the real number line, such that \( u \in U \) and \( v \in V \) are neighbors if and only if their corresponding intervals intersect.

For the following definition, we consider a biclique of a given graph to be a maximal bipartite-complete induced subgraph.

**Definition 4.** A bipartite graph \( B \) is a Helly circular-arc bigraph if it admits a Helly bi-circular-arc model. A bi-circular-arc model \((C, I, E)\) is Helly if, for any biclique in the graph it represents, there is a point \( p \in C \) such that all arcs corresponding to vertices of the biclique contain \( p \). A bipartite graph \( B \) is a Helly interval bigraph if it admits a Helly bi-interval representation. A bi-interval representation is Helly if, for any biclique of the graph it represents, the intervals corresponding to the vertices of the biclique all contain a common point \( p \) in the real number line.

It is easy to verify that both classes are hereditary over induced subgraphs, and that Helly interval bigraphs are a proper subclass of Helly circular-arc bigraphs. Lemma 4 presents a sufficient condition for a graph to be a Helly circular-arc bigraph. We denote by \((C_{2k}, S_{\ell})\) the graph obtained by adding \( \ell \) isolated vertices to a \( C_{2k} \).

**Lemma 4.** Let \( G \) be a bipartite graph. If \( G \) is \( K_{1,3}\)-free and \((C_{2k}, S_{\ell})\)-free (with \( k > 2 \)), then \( G \) is a Helly circular-arc bigraph.
Lemma 4 is a simple sufficient condition for the Helly subclass, serving to prune the search space for forbidden structures. It is known to not be necessary, since \( K_{1,3} \) by itself is a Helly circular-arc bigraph.

References


