Results on Circular-Arc Bigraphs

(Extended Abstract)

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Abstract. *We present a series of results related to the structural properties of the bipartite graph class known as circular-arc bigraphs. We also propose the definition of a Helly circular-arc bigraph subclass, based on a concept known as bipartite-Helly, along with a few results related to its structural properties.*

1. Introduction

A *circular-arc graph* is a graph that admits an intersection model of arcs on a circle. Arising as a generalization of *interval graphs*, the class has been extensively studied by many researchers since 1964 [Lin and Szwarcfiter 2009], yielding a plethora of relevant results. Trotter and Moore [Trotter and Moore 1976] presented infinite sets of minimally non circular-arc graphs. Francis et al. [Francis et al. 2014] presented a characterization of circular-arc graphs, along with a certifying recognition algorithm. Furthermore, Hell and Huang [Hell and Huang 2004] characterized *two-clique* circular arc graphs as the complements of *interval bigraphs*. Soulignac, in his thesis [Soulignac 2010], characterized different circular-arc subclasses, including the *Helly* subclass, based on the concept of *Helly families* [Helly 1923]. A linear-time algorithm for recognition of the class was presented by McConell [McConnell 2003], based on the so-called *circular-arc matrices*.

The concept of *circular-arc bigraphs* arises as a bipartite variation of the circular-arc concept, similar to interval bigraphs with relation to interval graphs. We treat circular-arc bigraph representations as *bi-circular-arc models* $\mathcal{B} = (C, \mathbb{I}, \mathbb{E})$, where C is a circle, and $\mathbb{I}$, $\mathbb{E}$ are families of arcs over C. It is then said that $\mathcal{B}$ represents a bipartite graph $G = (U, V, E)$ if there is a one-to-one correspondence between U and $\mathbb{I}$, and one between V and $\mathbb{E}$, such that, for every $u \in U$ and every $v \in V$, $\{u, v\} \in E$ if and only if the arc corresponding to u in $\mathbb{I}$ and the one corresponding v in $\mathbb{E}$ intersect.

Being a rather recent subject, it has yet to be as extensively studied as the other classes cited. Basu et al. [Basu et al. 2013] made a series of characterizations of circular-arc bigraphs, as well as the *proper* and *unit* subclasses, based on the so-called *biadjacency matrices*. Das et al. [Das and Chakraborty 2015] found a forbidden structure characterization of proper circular-arc bigraphs and interval bigraphs.

In this paper, we present a handful of results about the structural properties of circular-arc bigraphs, as well as the Helly subclass, which we introduce. Our study's main focus were the structural properties of circular-arc bigraphs, based on their graph structures instead of matrix structures. A handful of potentially useful results were found, which we present in the next section.

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2. Results on Circular-Arc Bigraphs

To represent circular arcs on a circle C of length n, we represent each point as a real number $0 \leq r < n$, which denotes a clockwise offset through the circle from a fixed zero point in C. It is then possible to denote a circular arc by a pair (r_1, r_2) of real numbers, where the arc is traced clockwise from point r_1 to point r_2, respectively called the counter-clockwise and clockwise endpoints of the arc. Furthermore, in the enunciation of the following results, every index sum is circular, which means that in an enumerated set $\{a_1, ..., a_n\}$, we have $a_{n+1} = a_1$ and $a_{1-1} = a_n$.

Lemma 1. *Let $B = (U, V, E)$ be a bipartite graph, with $|V| = n$ and $|U| = m$. If there is an ordering $S = (v_1, ..., v_n)$ of V such that, for every $u \in U$, the neighborhood of u is circularly consecutive with relation to S, then B is a circular-arc bigraph.*

Proof. Start with a circle C of length $n + 1$. For each $v_i \in V$, draw an arc $(i, i + 1) \in \mathbb{I}$ to represent it. Then, for each $u \in U$ whose neighborhood ranges from v_i to v_j, we draw an arc $(i + \frac{1}{2}, j + \frac{1}{2}) \in \mathbb{E}$ to represent it. $(C, \mathbb{I}, \mathbb{E})$ corresponds to B. □

Lemma 1 implies that graphs such as *generalized crowns* and *bipartite permutation graphs* are all circular-arc bigraphs.

Definition 1. *Let $a > 0$ and $0 \leq b < a$. The* generalized crown *graph S_b^a is a bipartite graph with vertex set $\{u_0, ..., u_{a-1}\} \cup \{v_0, ..., v_{a-1}\}$, such that the neighborhood of v_i is $U - \{u_i, ..., u_{i+b}\}$.*

Definition 2. *Let $G = (V, E)$ be a graph. Then G is a* permutation graph *if there exist two permutations P_1, P_2 of its vertex set such that, if the index of v in P_1 is less than that of w, then $\{v, w\} \in E$ if and only if the index of v in P_2 is greater than that of w. A bipartite permutation graph is simply a permutation graph that is bipartite.*

Let $\mathbb{A}$ be a family of arcs on a circle C. An arc B of C is said to *minimally intersect* $A \in \mathbb{A}$ if it intersects A without intersecting another arc $A' \in \mathbb{A}$ such that $A' \subset A$. Lemma 2 allows us to describe a type of forbidden structure for the class.

Lemma 2. *Let $0 < m < n$, and let $\mathbb{A}$ be a family of arcs on a circle C, with $|\mathbb{A}| = n$, such that no two arcs have coinciding endpoints. There are at most n subfamilies $\mathbb{A}' \subset \mathbb{A}$ such that $|\mathbb{A}'| = m$ for which it is possible to draw an arc on C that intersects every arc of $\mathbb{A}'$ without intersecting any of $\mathbb{A} - \mathbb{A}'$.*

Proof. Let B be an arc on C ($B \notin \mathbb{A}$) that intersects exactly m arcs of $\mathbb{A}$. Let $\mathbb{A}'$ be the family of arcs intersected by B, and let $S = (A_1, ..., A_m)$ be the order in which the counter-clockwise endpoint of each arc in $\mathbb{A}'$ is first encountered by trailing the circle clockwise from the clockwise endpoint of B. Let A_i be the first arc in the order S that is minimally intersected by B. We shall call A_i the *first minimally intersected arc* of B. Notice that $\mathbb{A}'$ contains all arcs that contain A_i. Let $k > 0$ be the number of such arcs. In particular, notice that every arc in $\{A_1, ..., A_{i-1}\}$ contains A_i, otherwise, either they would be minimally intesercted themselves, or they'd contain another arc that is minimally intersected and comes before A_i in S. The other arcs that $\mathbb{A}'$ contains will be exactly the first $m - k$ arcs that are not contained in (nor contain) A_i and whose counter-clockwise endpoints come immediately after the counter-clockwise endpoint of A_i. Therefore, if A_i is the first minimally intersected arc of a new arc B that intersects m arcs, $\mathbb{A}'$ will always be the same. That implies that for each arc $A \in \mathbb{A}$, there is at most one subfamily of m arcs which can be intersected by a new arc with A being the first minimally intersected arc (there might also be none, for instance, if an arc is contained in more than $m - 1$ arcs). Therefore, there are at most n subfamilies of m arcs that can be exclusively intersected by a new arc. □

Corollary 3. *Let $B = (U, V, E)$ be a bipartite graph, such that $|U| = n$, $|V| = n + 1$, and there is a k such that $|N(v)| = k$ for all $v \in V$. If the neighborhoods of the vertices of V are all pairwise distinct, then B is not a circular-arc bigraph.*

Figure 1 contains a handful of forbidden subgraphs we found for the class.

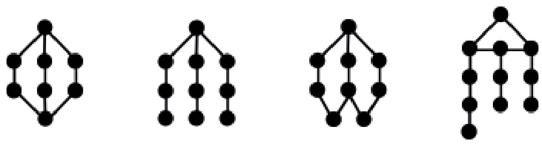

Figure 1. List of forbidden subgraphs we found for the circular-arc bigraph class.

2.1. Studies on the Helly Subclass

The concept of Helly families [Helly 1923] lends itself to the definition of many extensively studied graph classes [Szwarcfiter 1997, Groshaus and Szwarcfiter 2007, Groshaus and Szwarcfiter 2008]. An adaptation of the definition of Helly families, known as *bipartite-Helly*, was formulated by Groshaus and Szwarcfiter [Groshaus and Szwarcfiter 2010], and allows us to define Helly-like properties between pairs of families. For the results in this section, we first introduce *interval bigraphs* in Definition 3, and then *Helly circular-arc bigraphs* and *Helly interval bigraphs* in Definition 4, loosely based on the bipartite-Helly concept.

Definition 3. *A bipartite graph $B = (U, V, E)$ is an* interval bigraph *if it admits a bi-interval representation. A bi-interval representation of B is a mapping between $U \cup V$ and a family of intervals on the real number line, such that $u \in U$ and $v \in V$ are neighbors if and only if their corresponding intervals intersect.*

For the following definition, we consider a *biclique* of a given graph to be a maximal bipartite-complete induced subgraph.

Definition 4. *A bipartite graph B is a* Helly circular-arc bigraph *if it admits a Helly bi-circular-arc model. A bi-circular-arc model $(C, \mathbb{I}, \mathbb{E})$ is Helly if, for any biclique in the graph it represents, there is a point $p \in C$ such that all arcs corresponding to vertices of the biclique contain p. A bipartite graph B is a* Helly interval bigraph *if it admits a Helly bi-interval representation. A bi-interval representation is Helly if, for any biclique of the graph it represents, the intervals corresponding to the vertices of the biclique all contain a common point p in the real number line.*

It is easy to verify that both classes are hereditary over induced subgraphs, and that Helly interval bigraphs are a proper subclass of Helly circular-arc bigraphs. Lemma 4 presents a sufficient condition for a graph to be a Helly circular-arc bigraph. We denote by (C_{2k}, S_ℓ) the graph obtained by adding ℓ isolated vertices to a C_{2k}.

Lemma 4. *Let G be a bipartite graph. If G is $K_{1,3}$-free and (C_{2k}, S_ℓ)-free (with $k > 2$), then G is a Helly circular-arc bigraph.*

Proof. If the graph contains a C_{2n}, for $n > 2$, and doesn't contain any induced $K_{1,3}$ nor any isolated vertices, then its only component is the C_{2n}, which can easily be verified to be a Helly circular-arc bigraph. Now suppose the graph does not contain a C_{2n}, for $n > 2$. Then, since the graph is $K_{1,3}$-free, every single one of its components is either a path or a $K_{2,2}$. It is easy to verify that both paths and the $K_{2,2}$ are Helly interval bigraphs, meaning that a graph that has those graphs as connected components also is. □

Lemma 4 is a simple sufficient condition for the Helly subclass, serving to prune the search space for forbidden structures. It is known to not be necessary, since $K_{1,3}$ by itself is a Helly circular-arc bigraph.

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