The Geodesic Carathéodory Number

Eduardo S. Lira[1], **Diane Castonguay**[1], **Erika M. M. Coelho**[1], **Hebert Coelho**[1]

[1]Instituto de Informática – Universidade Federal de Goiás (UFG)
Alameda Palmeiras, Quadra D, Câmpus Samambaia – 74.690-900 – Goiânia – GO – Brazil

{eduardolira,diane,erikamorais,hebert}@inf.ufg.br

Abstract. *From Carathéodory's theorem arises the definition of the Carathéodory number for graphs. This number is well-known for monophonic and Triangle-path convexities. It is limited for some classes of graphs on P_3 and geodesic convexities but is known to be unlimited only on P_3-convexity. In this paper, we prove that the Carathéodory number is unlimited on geodesic convexity.*

Resumo. *Do teorema de Carathéodory surge a definição do número de Carathéodory para grafos. Este número é bem conhecido nas convexidades monofônica e de caminho de triângulos. Ele é limitado para algumas classes de grafos nas convexidades P_3 e geodésica, mas apenas na convexidade P_3 sabe-se que ele é ilimitado. Neste artigo, nós provamos que o número de Carathéodory é ilimitado na convexidade geodésica.*

1. Introduction

In 1911, Constantin Carathéodory published a theorem stating that every u in the convex hull of a subset $S \subseteq R^d$, is also in the convex hull of a subset F of S of order at most $d + 1$ [Carathéodory 1911] . The ideas provenient from this theorem, brought to light an invariant in graph theory, called Carathéodory number, widely studied since then. We will focus on the study of this number on geodesic convexity.

In this article, we used the following definition for the Carathéodory number of a graph. A subset S of vertices is a *Carathéodory set* if there exists a vertex in the convex hull of S which does not belong to the convex hull of the $S\backslash\{s\}$ for every $s \in S$. The *Carathéodory number* $c(G)$ of a graph G is the maximum cardinality of a Carathéodory set [E. M. M. Coelho and Szwarcfiter 2013].

The Carathéodory number is well-known for some convexities. On monophonic convexity, $c(G) = 1$ for complete graphs, and 2 for other graphs [Duchet 1988]. On Triangle-path convexity $c(G) = 2$ [Changat and Mathew 1999]. On P_3-convexity, $c(G) \leq 3$ for multipartite tournaments [D.B. Parker and Wolf 2008], however, it is unlimited considering general graphs [E. M. M. Coelho and Szwarcfiter 2013]. Considering the geodesic convexity, [M.C. Dourado 2013] showed that it is NP-complete to determine whether $c(G)$ is at least a given k, even when restricted to bipartite graphs, and that $c(G)$ is at most 3 for split graphs.

This article brings a new result on this matter: the Carathéodory number is unlimited on geodesic convexity.

2. Preliminaries

Let G be a finite graph given by a set of vertices $V(G)$ and a set of edges $E(G)$. A set $\mathcal{C}$ of subsets of $V(G)$ is a *convexity* if $\emptyset, V(G) \in \mathcal{C}$ and $\mathcal{C}$ is closed under intersections. The elements of $\mathcal{C}$ are called *convex sets*. Given a set $S \subset V$, the *convex hull* of S is the smallest convex set $H(S) \in \mathcal{C}$, such that $S \subseteq H(S)$.

Several convexities are defined by a set $\mathcal{P}$ of paths in a graph. In this scenario, a subset $\mathcal{C} \in V(G)$ is convex when for all $x, y \in \mathcal{C}$, if $p \in \mathcal{P}$ is a path between x and y, then all vertices in p are also in $\mathcal{C}$.

The *geodesic convexity* is obtained when $\mathcal{P}$ is the set of all geodesics, i.e., all the shortest paths. A *chord* of a path p in a graph G is any edge joining a pair of nonadjacent vertices of P. The *monophonic convexity* is defined using the set of all chordless paths, i.e., induced paths. The *triangle-path convexity* arise from the set of all paths in which every chord is a *short chord*, i.e., chords joining vertices at distance 2 apart. And the P_3-*convexity*, also known as *two-path convexity*, is obtained from the set of all paths of length 2.

The *Carathéodory number* of a graph G, written as $c(G)$, is the smallest integer k such that for every subset S of $V(G)$ and every element s in $H(S)$, there is a subset F of S with $|F| \leq k$ and $u \in H(F)$.

A subset $S \subset V(G)$ is called *Carathéodory set* when $\partial H(S) = H(S) \setminus \bigcup_{s \in S} H(S \setminus \{s\})$ is nonempty. We can also define the Carathéodory number as the maximum cardinality of a Carathéodory set [E. M. M. Coelho and Szwarcfiter 2013].

3. Geodesic Carathéodory Number is Unlimited

We now construct inductively a family of graphs G_i, with $i \geq 1$ that have an unlimited Carathédory number on geodesic convexity:

- G_1 is the graph with $V(G_1) = \{v_1\}$, and $E(G_1) = \emptyset$;
- G_2 is the graph with $V(G_2) = V(G_1) \cup \{v_2, v_3\}$, and $E(G_2) = E(G_1) \cup \{v_1v_3, v_2v_3\}$;
- G_3 is the graph with $V(G_3) = V(G_2) \cup \{v_4, v_5, v_6\}$, and $E(G_3) = E(G_2) \cup \{v_1v_4, v_2v_4, v_3v_4, v_3v_5, v_4v_6, v_5v_6\}$;
- G_{i+1}, for $i \geq 3$, is the graph with $V(G_{i+1}) = V(G_i) \cup \{v_{n-2}, v_{n-1}, v_n\}$, and $E(G_{i+1}) = E(G_i) \cup \{v_{n-4}v_{n-2}, v_{n-3}v_{n-2}, v_{n-2}v_{n-1}, v_{n-1}v_n, v_4v_n\}$ where $n = 3i$.

Theorem 1. *Every G_i, constructed as above, has a Carathéodory set of cardinality i.*

Proof. For $i = 1$, $S_1 = \{v_1\}$ is a Carathéodory set of G_1. For $i = 2$, $S_2 = \{v_1, v_2\}$ and $v_3 \in \partial H(S_2)$. We will now show by induction that $S_i = \{v_1, v_2\} \cup \{v_{3j} \mid 2 \leq j \leq i\}$ is a Carathéodory set of G_i, with cardinality i, for $i \geq 3$. For $i = 3$, note that $v_5 \in \partial H(S_3)$. For $i = 4$, we have that $v_7, v_8 \in \partial H(S_4)$ and for $i = 5$, one can see that $\partial H(S_5) = \{v_{10}, v_{11}\}$. This proves that S_i is a Carathéodory set of G_i, with cardinality i, for $1 \leq i \leq 5$.

Take $i \geq 5$ and assume (as inductive hypothesis) that the longest geodesic of G_i is at most 4, and S_i is a Carathéodory set of cardinality i, with $\{v_{n-5}, v_{n-4}\} = \partial H(S_i)$, where $n = 3i$.

Take $S_{i+1} = S_i \cup \{v_n\}$, and note that all vertices in S_{i+1} are connected to v_4, and that any other vertex in $V(G_{i+1})$ is connected to at least one vertex in S_{i+1}, by construction of G_{i+1}. Therefore, the longest geodesic of G_{i+1} is at most 4. There are three edges connecting the vertices in $V(G_i)$ to those new vertices added to build G_{i+1}, namely $v_{n-4}v_{n-2}$, $v_{n-3}v_{n-2}$, and v_4v_n. Choose any two vertices x, y in $V(G_i)$, other than v_4 (note that the geodesic distance of any vertex to v_4 is at most 2). If you take a path between x and y that uses both edges $v_{n-4}v_{n-2}$ and $v_{n-3}v_{n-2}$, it is bigger then their geodesic in at least one unit, since $v_{n-4}v_{n-3} \in E(G_i)$. If you take a path between x and y that uses $v_{n-4}v_{n-2}$ or $v_{n-3}v_{n-2}$, and v_4v_n, this path is bigger than 4. So, by inductive hypothesis, in G_{i+1} no shorten path was introduced between two vertices of G_i, implying that $H(S_i)$ in G_{i+1} is exactly $V(G_i)$.

All the geodesics between v_n and another vertex in S_{i+1} have size 2, and between v_n and another vertex not in S_{i+1} have size at most 3. Take any vertex in $V(G_i)$ that is not in S_{i+1}, other than v_{n-4}. If a path between such vertex and v_n uses either edge $v_{n-4}v_{n-2}$ or $v_{n-3}v_{n-2}$, it is bigger than 3, so it is not a geodesic. The geodesics between v_{n-4} and v_n are $v_{n-4}v_{n-2}v_{n-1}v_n$ and $v_{n-4}v_{n-3}v_4v_n$. This means the only minimum path between a vertex in $V(G_i)$ and v_n that uses the vertices v_{n-2} and v_{n-1} is the one between v_{n-4} and v_n. Without loss of generality, using the inductive hypothesis, we can conclude that $\partial H(S_{i+1}) = \{v_{n-2}, v_{n-1}\}$ (meaning S_{i+1} is a Carathéodory set), and $|S_{i+1}| = i + 1$, completing the proof.

□

Corollary 1. *The Carathéodory number on geodesic convexity is unlimited.*

Proof. By Thereom 1, for all integer $i \geq 1$, there is a Carathéodory set of cardinality i in G_i. We conclude that $c(G_i) \geq i$ and thus yields the results. □

References

Carathéodory, C. (1911). Uber den variabilit$\ddot{a}$tsbereich der fourierschen konstanten von positiven harmonischen funktionen. In *Rend. Circ. Mat. Palermo*, pages 193 – 217.

Changat, M. and Mathew, J. (1999). On triangle path convexity in graphs. In *Discrete Math*, pages 91–95.

D.B. Parker, R. W. and Wolf, M. (2008). On two-path convexity in multipartite tournaments. In *European J. Combin.*, pages 641–651.

Duchet, P. (1988). Convex sets in graphs ii. minimal path convexity. In *J. Combin Theory, Ser. B*, pages 307–316.

E. M. M. Coelho, M.C. Dourado, D. R. and Szwarcfiter, J. (2013). The carathéodory number of the p_3 convexity of chordal graphs. In *The Seventh European Conference on Combinatorics, Graph Theory and Applications*, pages 209–214.

M.C. Dourado, D. Rautenbach, e. a. (2013). On the carathéodory number of interval and graph convexities. In *Theoretical Computer Science*, pages 127–135.