

# Degree-preserving companion of Nelson logic expanded with a consistency operator

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**Abstract.** *The main aim of this paper is defining a Logic of Formal Inconsistency over the degree-preserving companion of Nelson logic with a consistency operator. In this sense, we present a quasivariety of Nelson lattices enriched with a suitable consistency operator and axiomatise the corresponding logic. As main results we present necessary and sufficient conditions to prove a categorial equivalence for the category for Nelson lattices with a consistency operator.*

**Resumo.** *O principal objetivo desse artigo é definir uma Lógica da Inconsistência Formal sobre a lógica que preserva gaus de verdade que acompanha a lógica de Nelson com um conectivo de consistência. Nesse sentido, apresentamos uma quase-variedade de reticulados de Nelson enriquecidos com um operador de consistência adequado e axiomatizamos a lógica correspondente. Como resultado principal, apresentamos as condições necessárias e suficientes para demonstrar uma equivalência categorial para os reticulados de Nelson com um operador de consistência.*

## 1. Introduction

In the 1950's, Constructive logic with strong negation (CLSN) was formulated independently by Nelson and Markov as a result of certain philosophical objections to intuitionistic negation. The criticism of intuitionistic negation concerns its disadvantageous non-constructive property, namely, that the derivability of the formula  $\neg(\alpha \wedge \beta)$  in an intuitionistic propositional calculus does not imply that at least one of the formulas  $\neg\alpha$  or  $\neg\beta$  is derivable.

More recently, in 2008, Spinks and Veroff have shown CLSN, also known in the literature as Nelson logic, can be considered as a substructural logic. More precisely, they have shown that the algebraic models of CLSN, *Nelson algebras*, are termwise equivalent to certain involutive, bounded, commutative and integral residuated lattices, called *Nelson (residuated) lattices*, see [6].

The main aim of this paper is defining a Logic of Formal Inconsistency (LFI) based on Nelson logic, by considering the degree-preserving companion of Nelson logic expanded with a consistency operator, in the style of [4]. To do so, we will first algebraically study the quasivariety of Nelson lattices with a consistency operator, then we look at a categorial equivalence between Heyting algebras (with extra structure) and Nelson lattices with consistency operators, and finally we define and axiomatise the logic.

## 2. Preliminaries

Nelson algebras were introduced by Rasiowa (see, for instance, [7]), under the name of N-lattices, as the algebraic counterpart of the constructive logic with strong negation considered by Nelson and Markov. A Nelson algebra is an algebra  $\mathbf{N} = (N, \vee, \wedge, \Rightarrow, \neg, \top, \perp)$  of type  $(2, 2, 2, 1, 0, 0)$  such that  $(N, \vee, \wedge, \neg, \top, \perp)$  is a Kleene algebra, and the implication  $\Rightarrow$  satisfies the following equations:

$$\begin{aligned} x \Rightarrow x &= \top & x \Rightarrow (y \wedge z) &= (x \Rightarrow y) \wedge (x \Rightarrow z) \\ x \wedge (x \Rightarrow y) &= x \wedge (\neg x \vee y) & x \Rightarrow (y \Rightarrow z) &= (x \wedge y) \Rightarrow z \end{aligned}$$

This definition of Nelson algebras is due to Brignole and Monteiro, and provides an equational characterization of the N-lattices introduced by Rasiowa.

On the other hand, as mentioned above, in [6], the authors have shown that Nelson algebras are termwise equivalent to a certain class of *involutive residuated lattices*, called Nelson residuated lattices or Nelson lattices. Recall that a *commutative, integral, bounded residuated lattice*, that we will simply call *residuated lattice*, is an algebra  $\mathcal{A} = \langle A, \wedge, \vee, *, \rightarrow, \perp, \top \rangle$  of type  $(2, 2, 2, 2, 0, 0)$  such that  $\langle A, *, \top \rangle$  is a commutative monoid,  $\mathcal{L}(A) = \langle A, \wedge, \vee, \perp, \top \rangle$  is a bounded lattice with least element  $\top$  and greatest element  $\perp$ , and such that the following condition holds:  $x * y \leq z$  iff  $x \leq y \rightarrow z$ , where  $x, y, z$  denote arbitrary elements of  $A$  and  $\leq$  is the order given by the lattice structure. Since we assume the neutral element of the monoid reduct coincides with the greatest element of  $\mathcal{L}(A)$  we have that:  $x \leq y$  iff  $x \rightarrow y = \top$ .

It is well-known that the class **RL** of residuated lattices is a variety, which is related to different and well-known varieties studied in substructural and fuzzy logics literature. In fact **RL** coincides with the variety of  $\mathbf{FL}_{ew}$ -algebras of [5]. We use the denotational conventions of [5]: **FL** refers to the ‘‘Full Lambek calculus’’, which is the base system and associated algebras, and subindices indicate several axiomatic extensions with properties such as exchange ( $e$ ) or weakening ( $w$ ).

A residuated lattice is called *involutive* if it satisfies the double negation equation:  $\neg\neg x = x$ , where  $\neg x$  is  $x \rightarrow \perp$ . Besides, it is possible to prove that:  $x * y = \neg(x \rightarrow \neg y)$  and  $x \rightarrow y = \neg(x * \neg y)$ .

A *Nelson residuated lattice*, or simply *Nelson lattice*, is an involutive residuated lattice that satisfies the following identity:

$$(((x * x) \rightarrow y) \wedge ((\neg y * \neg y) \rightarrow \neg x)) \rightarrow (x \rightarrow y) = \top.$$

Nelson lattices will be also called in the paper **NL**-algebras.

In [2] the authors show that, given a Nelson algebra  $\mathbf{N} = (N, \vee, \wedge, \Rightarrow, \neg, \top, \perp)$ , then the algebra  $\mathcal{R}(\mathbf{N}) = (N, \wedge, \vee, *, \rightarrow, \perp, \top)$  is a Nelson residuated lattice, where the operations  $*$  and  $\rightarrow$  are defined as follows:

$$x * y := \neg(x \Rightarrow \neg y) \vee \neg(y \Rightarrow \neg x), \quad x \rightarrow y := (x \Rightarrow y) \wedge (\neg y \Rightarrow \neg x),$$

and moreover it holds  $\neg x = x \rightarrow \perp$  for each  $x \in N$ . Conversely, consider now a Nelson lattice  $\mathbf{A} = \langle A, \wedge, \vee, *, \rightarrow, \perp, \top \rangle$  and define a binary operation  $\rightarrow_N$  by specifying  $x \rightarrow_N y := (x * x) \rightarrow y$ . Then,  $\mathcal{N}(A) = \langle A, \wedge, \vee, \rightarrow_N, \neg, \perp, \top \rangle$  is a Nelson algebra.

For a given a Heyting algebra  $\mathbf{H}$  and a Boolean filter  $F$  of  $\mathbf{H}$ , let us define  $N(H, F) := \{(x, y) \in H \times H : x \wedge y = \perp, x \vee y \in F\}$ . Then we have that

$\mathbf{N}(\mathbf{H}, F) = (N(H, F), \vee, \wedge, \Rightarrow, \neg, * \perp, \top)$  is a Nelson lattice, where the operations are defined as follows:

$$\begin{aligned}(x, y) \vee (s, t) &= (x \vee s, y \wedge t), \\ (x, y) \wedge (s, t) &= (x \wedge s, y \vee t), \\ (x, y) * (s, t) &= (x \wedge s, (x \rightarrow t) \wedge (s \rightarrow y)), \\ (x, y) \Rightarrow (s, t) &= ((x \rightarrow s) \wedge (t \rightarrow y), x \wedge t), \\ \neg(x, y) &= (y, x), \\ \top &= (\top, \perp), \\ \perp &= (\perp, \top).\end{aligned}$$

Finally, we briefly recall the notion a *Logic of Formal Inconsistency*, LFI for short, see e.g. [3]. Let  $\Sigma'$  be a propositional signature and assume a denumerable set  $\mathcal{V} = \{p_1, p_2, \dots\}$  of propositional variables. The propositional language generated by  $\Sigma'$  from  $\mathcal{V}$  will be denoted by  $\mathcal{L}_{\Sigma'}$ . On the other hand, let  $\mathbf{L} = \langle \Sigma', \vdash \rangle$  be a Tarskian, finitary and structural logic defined over the propositional signature  $\Sigma'$ , which contains a negation  $\neg$ , and let  $\circ$  be a primitive or defined unary connective. Then,  $\mathbf{L}$  is said to be a *Logic of Formal Inconsistency with respect to  $\neg$  and  $\circ$*  if the following holds:

- (i)  $\varphi, \neg\varphi \not\vdash \psi$ , for some  $\varphi$  and  $\psi$ ;
- (ii) there are two formulas  $\alpha$  and  $\beta$  such that  $\circ\alpha, \alpha \not\vdash \beta$  and  $\circ\alpha, \neg\alpha \not\vdash \beta$ ;
- (iii)  $\circ\varphi, \varphi, \neg\varphi \vdash \psi$ , for every  $\varphi$  and  $\psi$ .

### 3. Nelson residuated lattices with a consistency operator $\circ$

We start by formally defining the class of Nelson lattices with a consistency operator.

**Definition 3.1** *NL $_{\circ}$ -algebras are expansions of Nelson lattices with a new unary operation  $\circ$  satisfying the following equation and quasi-equation:*

- (o1)  $x \wedge \neg x \wedge \circ(x) = 0$
- (o2) *if  $x \wedge \neg x \wedge y = 0$  then  $y \leq \circ(x)$*

In the following, we will use the expression  $(\mathbf{A}, \circ)$  to denote the NL $_{\circ}$ -algebra whose Nelson lattice reduct is  $\mathbf{A}$ .

From this definition, it is clear that the class of NL $_{\circ}$ -algebras is a quasivariety. Now, we have the following easy properties of NL $_{\circ}$ -algebras are displayed in the next lemma.

**Lemma 3.2** *The following properties hold in a NL $_{\circ}$ -algebra  $(\mathbf{A}, \circ)$ :*

- (i)  $\circ x = \max\{z \in A \mid x \wedge \neg x \wedge z = 0\}$
- (ii)  $\circ(x) = \circ(\neg x) = \circ(x \wedge \neg x) = \circ(x \vee \neg x)$
- (iii)  $\circ(1) = \circ(0) = 1$

It follows from (i) of Lemma 3.2 that if a consistency operator is definable in a Nelson lattice it is uniquely determined. Moreover, it also tells us that a consistency operator is always definable in every finite Nelson lattice. However it is not always the case in infinite Nelson lattices as we can see in the next example.

**Example 3.3** Let  $\mathbf{A}$  be the Nelson lattice over  $[0, 1] \setminus \{\frac{1}{2}\}$  with the Nilpotent Minimum operations:  $x * y = \min(x, y)$  if  $x + y > 1$  and  $x * y = 0$  otherwise; and  $\neg x = 1 - x$ . Let  $\mathbf{B}$  be the subalgebra of  $\mathbf{A} \times \mathbf{A}$  defined on the sublattice consisting of the elements of the form  $(x, y)$  with  $x, y$  being positive elements of  $\mathbf{A}$  (i.e. such that  $1 \geq x, y > \frac{1}{2}$ ) and their negations  $(1 - x, 1 - y)$ . Take an element  $(x, 1) \in B$  such that  $1 > x > \frac{1}{2}$ . An easy computation shows that  $\circ((x, 1))$  does not exist.

As is well-known, an implicative filter of a bounded residuated lattice  $A$  is a subset  $F \subseteq A$  such that  $\top \in F$  and it is closed under modus ponens:  $x \in F$  and  $x \rightarrow y \in F$  imply  $y \in F$ . For each implicative filter  $F$ , the binary relation  $\Theta(F)$  defined by  $(x, y) \in \Theta(F)$  if and only if  $x \rightarrow y, y \rightarrow x \in F$  is a congruence of the residuated lattice  $A$ , and  $F = \{z \in A : (z, \top) \in \Theta(F)\}$ . This is actually a one-one correspondence between the lattice of congruences and the lattice of implicative filters for the variety of bounded residuated lattices. However, since the class of expanded Nelson lattices with a consistency operator involves not only equations but also a quasi-equation, it is a quasivariety. In a quasivariety, congruences that allow for the decomposition of an algebra as a subdirect product of subdirectly irreducible components are required to satisfy an additional condition: the quotient of an algebra by a congruence has to belong to the quasivariety. This condition is automatically satisfied in varieties but not in quasivarieties. Congruences satisfying this condition are usually called Q-congruences. Similarly, filters that are in a one-one correspondence between Q-congruences are implicative filters ‘closed’ by the quasiequations of the quasivariety, and are called Q-filters.

Therefore, a filter  $F$  of a  $NL_{\circ}$ -algebra, besides being implicative, has to additionally satisfy the following two conditions:

- (F1) if  $x \rightarrow y, y \rightarrow x \in F$  then  $\circ x \rightarrow \circ y, \circ y \rightarrow \circ x \in F$
- (F2) if  $x \vee \neg x \vee \neg y \in F$  then  $y \rightarrow \circ x \in F$ .

We shall call such a filter a  $\circ$ -filter. Note that from (F1) it follows in particular that  $\circ$ -filters are closed by  $\circ$ : if  $x \in F$  then  $\circ x \in F$  as well.

**Lemma 3.4** If  $(\mathbf{A}, \circ)$  is  $NL_{\circ}$ -algebra such that  $\mathbf{A}$  is a subdirectly irreducible Nelson lattice, then  $\circ$  is such that  $\circ(1) = \circ(0) = 1$  and  $\circ(x) = 0$  otherwise, and hence  $(\mathbf{A}, \circ)$  is a simple  $NL_{\circ}$ -algebra.

In the following, if  $\mathbf{A}$  is a Nelson lattice, we will denote by  $B(\mathbf{A})$  the set of its Boolean elements, i.e.  $B(\mathbf{A}) = \{x \in A \mid x \wedge \neg x = 0\}$ .

**Theorem 3.5** Let  $(\mathbf{A}, \circ)$  be a subdirectly irreducible  $NL_{\circ}$ -algebra. Then  $B(\mathbf{A}) = \{0, 1\}$ .

When the algebra is finite, then we can prove more.

**Theorem 3.6** Let  $(\mathbf{A}, \circ)$  be a finite  $NL_{\circ}$ -algebra. Then the following conditions are equivalent:

- (i)  $(\mathbf{A}, \circ)$  is a s.i.  $NL_{\circ}$ -algebra,
- (ii)  $B(\mathbf{A}) = \{0, 1\}$ ,
- (iii)  $(\mathbf{A}, \circ)$  is a simple  $NL_{\circ}$ -algebra.

**Theorem 3.7** Let  $(\mathbf{A}, \circ)$  be a  $NL_{\circ}$ -algebra, then the following conditions are equivalent:

- (i)  $(\mathbf{A}, \circ)$  is subdirectly irreducible but not simple,
- (ii)  $B(\mathbf{A}) = \{0, 1\}$  and there is a non-trivial  $\circ$ -filter  $F$  such that  $(\circ(x))^2 \neq 0$  for every  $x \in F$ .

## 4. A categorial equivalence

In this section we show an equivalence between a category that involves Heyting algebras (with extra structure) and the algebraic category of Nelson lattices with consistency operators.

For convenience, let us represent  $\mathbf{A}$  as  $\mathbf{N}(\mathbf{H}, F)$  for  $\mathbf{H}$  being a Heyting algebra and  $F$  being a Boolean filter of  $\mathbf{H}$ . So, every  $x \in A$  is of the form  $(a, b)$  for  $a, b \in H$  such that  $a \wedge b = \perp$  and  $(a \vee b) \in F$ . Thus, condition (i) of Lemma 3.2 which indeed characterizes the  $\circ$  in a Nelson lattice, can be reformulated in the following way: for all  $(a, b) \in A$ ,

$$o(a, b) = \max\{(z, z') \in A \mid (\perp, a \vee b \vee z') = (\perp, \top)\}$$

where the equality holds because  $a \wedge b = \perp$  (in  $H$ ) for all  $(a, b) \in A$ . The order on  $A$  is defined, with respect to the order of  $H$ , as  $(a, b) \leq (c, d)$  iff  $a \leq c$  and  $b \geq d$ .

**Lemma 4.1** *For every Heyting algebra  $\mathbf{H}$ , for every Boolean filter  $F$  of  $\mathbf{H}$  and for every  $a \in H$ ,  $a \vee \neg a \in F$ .*

Let us recall from that a unary operation  $\neg$  on a Heyting algebra  $\mathbf{H}$  is called a *dual pseudocomplement*, provided that the following equations are satisfied:

- (D1)  $x \vee \neg(x \vee y) = x \vee \neg y$ ,
- (D2)  $x \vee \neg 1 = x$ ,
- (D3)  $\neg\neg 1 = 1$ .

In every Heyting algebra  $\mathbf{H}$  and for every  $x \in H$ , if is the dual pseudocomplement of  $x$  exists, then it is defined as

$$\neg x = \min\{z \in H \mid x \vee z = 1\}. \quad (1)$$

**Proposition 4.2** *Let  $\mathbf{H}$  be a Heyting algebra,  $F$  a Boolean filter of  $\mathbf{H}$  and  $\mathbf{A}$  the Nelson lattice  $\mathbf{N}(\mathbf{H}, F)$ . If  $\neg(a \vee b)$  exists in  $H$ , then  $\circ(a, b)$  exists and  $\circ(a, b) = (\neg d, d)$ , where  $d = \neg(a \vee b)$ . Besides, for any  $(a, b) \in A$ ,  $\circ(a, b) \in A$ .*

**Corollary 4.3** *Let  $\mathbf{H}$  be a Heyting algebra,  $F$  a Boolean filter of  $\mathbf{H}$  and  $\mathbf{A}$  the Nelson lattice  $\mathbf{N}(\mathbf{H}, F)$ . Then the operation  $\circ$  exists in  $\mathbf{A}$  if and only if the dual pseudo-complement exists for all the elements of  $F$ . In particular, if  $\mathbf{A} = \mathbf{N}(\mathbf{H}, H)$ , then  $\circ$  exists in  $\mathbf{A}$  if and only if the dual pseudo-complementation is definable in the whole  $\mathbf{H}$ .*

### 4.1. A categorial equivalence for Nelson lattices with a consistency operator

To ease the notation, let us denote by  $BPF(\mathbf{H})$  the set of the Boolean filters  $F$  of a Heyting algebra  $\mathbf{H}$  that further satisfy the following property:

- (DP) For every  $x \in F$ ,  $\neg x$  exists in  $\mathbf{H}$ .

Thus,  $BPF(\mathbf{H})$  is the set of Boolean filters  $F$  of  $\mathbf{H}$  such that every element  $x \in F$  has a dual pseudocomplement in  $\mathbf{H}$ .

**Definition 4.4** *Consider the set  $\mathbf{HBP}$  containing pairs  $(\mathbf{H}, F)$  such that  $\mathbf{H}$  is a Heyting algebra,  $F \in BPF(\mathbf{H})$  and maps defined as follows: given two pairs  $(\mathbf{H}, F)$  and  $(\mathbf{H}', F')$  a map  $h$  between them is such that:*

- (m1)  $h$  is a Heyting homomorphism between  $\mathbf{H}$  and  $\mathbf{H}'$ ,

- (m2)  $h(F) \subseteq F'$ ,  
(m3) for all  $x \in F$ ,  $h(\neg x) = \neg' h(x)$ .

It is not difficult to see that **HBP** is a category and hence we will respectively call *objects* and *morphisms* the pairs  $(\mathbf{H}, F)$  and the maps  $h : (\mathbf{H}, F) \rightarrow (\mathbf{H}', F')$  of **HBP** defined as above.

Moreover, let **NC** be the algebraic category of Nelson lattices with a consistency operator as in Definition 3.1 and let us consider the map  $\mathcal{NC} : \mathbf{HBP} \rightarrow \mathbf{NC}$  defined in the following way:

- For every object  $(\mathbf{H}, F) \in \mathbf{HBP}$ ,  $\mathcal{NC}(\mathbf{H}, F) = (\mathbf{N}(\mathbf{H}, F), \circ)$ , where  $\mathbf{N}(\mathbf{H}, F)$  is the Nelson algebra of pairs  $(a, b) \in H \times H$  such that  $a \wedge b = \perp$  and  $a \vee b \in F$  as in the above section, and  $\circ(a, b) = (\neg\neg(a \vee b), \neg(a \vee b))$  as in Proposition 4.2.
- For every morphism  $h : (\mathbf{H}, F) \rightarrow (\mathbf{H}', F')$  of **HBP**,  $\mathcal{NC}(h) : \mathcal{NC}(\mathbf{H}, F) \rightarrow \mathcal{NC}(\mathbf{H}', F')$  is so defined: for all  $(a, b) \in \mathcal{NC}(\mathbf{H}, F)$ ,

$$\mathcal{NC}(h)(a, b) = (h(a), h(b)).$$

For every object  $(\mathbf{H}, F)$ ,  $\mathcal{NC}(\mathbf{H}, F)$  is an object in **NC**. Analogously, if  $h$  is a morphism of **HBP**,  $\mathcal{NC}(h)$  is a morphism of **NC**. The we can prove the following

**Proposition 4.5** *The map  $\mathcal{NC} : \mathbf{HBP} \rightarrow \mathbf{NC}$  is a functor.*

Recall that a functor between two categories yields a categorial equivalence iff it is full, faithful and essentially surjective.

**Theorem 4.6** *The functor  $\mathcal{NC}$  establishes a categorial equivalence between **HBP** and **NC**.*

## 5. Adding a consistency operator to Nelson logic and its paraconsistent companion

Let  $L$  be the logic of a variety of residuated lattices, that is, an axiomatic extension of  $FL_{ew}$ . For each such a logic  $L$ , in [1] the authors introduce a companion logic denoted  $L^{\leq}$ , whose associated consequence relation has the following semantics: for every set of formulas  $\Gamma \cup \{\varphi\}$ ,

$$\Gamma \vdash_{L^{\leq}} \varphi \quad \text{iff} \quad \text{for every } L\text{-algebra } \mathbf{A}, \text{ every } a \in A, \text{ and every } \mathbf{A}\text{-evaluation } e, \\ \text{if } a \leq e(\psi) \text{ for every } \psi \in \Gamma, \text{ then } a \leq e(\varphi).$$

For this reason  $L^{\leq}$  is known as the *degree-preserving companion* of  $L$ .

Now, let us consider the logic  $NL$  corresponding to the variety of Nelson lattices and its degree-preserving companion  $NL^{\leq}$ . Since the negation in  $NL$  is involutive, and hence it does not satisfy the pseudo-complementation axiom  $\varphi \wedge \neg\varphi \rightarrow \bar{0}$ , the logic  $NL^{\leq}$  is paraconsistent, i.e. in general

$$\varphi, \neg\varphi \not\vdash_{NL^{\leq}} \bar{0}$$

However,  $NL^{\leq}$  is not a *logic of formal inconsistency* (LFI) since we cannot define in the language of  $NL$  a consistency operator  $\circ$  satisfying  $\{\circ\varphi, \varphi, \neg\varphi\} \vdash \psi$  for every  $\varphi$  and  $\psi$ . Therefore, we consider next the expansion of  $NL$  with a suitable consistency operator  $\circ$ , that we will call  $NL_{\circ}$ , so that its degree-preserving companion  $NL_{\circ}^{\leq}$  is a LFI.

**Definition 5.1** We define the logic  $NL_\circ$  as the expansion of  $NL$  in a language which incorporates a new unary connective  $\circ$  with the following additional axioms and rules:

- (A1)  $\circ\bar{1}$
- (A2)  $\circ\bar{0}$
- (A3)  $\neg(\varphi \wedge \neg\varphi \wedge \circ\varphi)$

and the following inference rules:

$$\text{(CNG)} \quad \frac{\varphi \leftrightarrow \psi}{\circ\varphi \leftrightarrow \circ\psi} \quad \text{(Max)} \quad \frac{\varphi \vee \neg\varphi \vee \neg\psi}{\psi \rightarrow \circ\varphi}$$

**Lemma 5.2** The following derivabilities hold in  $NL_\circ$ :

i)  $\varphi \vee \neg\varphi \vdash_{L_\circ} \circ\varphi$ , (ii)  $L_\circ \vdash \circ\varphi \rightarrow \neg\varphi \vee \varphi$ , (iii)  $\varphi \vee \neg\varphi \dashv\vdash_{L_\circ} \circ\varphi$ .

It is easy to check that, due to the (Cong) rule for  $\circ$ ,  $NL_\circ$  is a Rasiowa implicative logic and hence it is algebraizable. The equivalent algebraic semantics is given by the quasi-variety of  $NL_\circ$ -algebras. As a direct consequence we have the following general completeness result.

**Proposition 5.3**  $NL_\circ$  is strongly complete w.r.t the class of  $NL_\circ$ -algebras.

**Definition 5.4** The degree-preserving companion of the logic  $NL_\circ$  is the logic  $NL_\circ^\leq$  defined by the following axioms and rules:

- Axioms of  $NL_\circ^\leq$  are those of  $NL_\circ$
- Rules of  $NL_\circ^\leq$  are:
  - (Adj- $\wedge$ ) from  $\varphi$  and  $\psi$  derive  $\varphi \wedge \psi$
  - (MP-r) if  $\vdash_{NL_\circ} \varphi \rightarrow \psi$ , then from  $\varphi$  and  $\varphi \rightarrow \psi$ , derive  $\psi$
  - (CNG-r) if  $\vdash_{NL_\circ} \varphi \leftrightarrow \psi$ , then from  $\varphi \leftrightarrow \psi$  derive  $\circ\varphi \leftrightarrow \circ\psi$
  - (Max-r) if  $\vdash_{NL_\circ} \neg(\varphi \wedge \neg\varphi \wedge \psi)$ , then from  $\neg(\varphi \wedge \neg\varphi \wedge \psi)$  derive  $\psi \rightarrow \circ\varphi$

It is clear that the degree-preserving companion  $NL_\circ^\leq$  is a LFI.

Let  $L$  be a paraconsistent logic with a consistency operator  $\circ$ . Then we say that  $\circ$  satisfies the *propagation property* in  $L$  with respect to a subset  $X$  of connectives of the language of  $L$  if  $\{\circ\varphi_1, \dots, \circ\varphi_n\} \vdash_L \circ\#\varphi_1, \dots, \varphi_n$ , for every  $n$ -ary connective  $\# \in X$  and formulas  $\varphi_1, \dots, \varphi_n$  built with connectives from  $X$ .

In particular, checking whether  $NL_\circ^\leq$  satisfies the propagation property for the connectives  $X = \{\bar{0}, \wedge, \&, \rightarrow\}$  amounts to check the following conditions:  $\vdash_{NL_\circ} \circ\bar{0}$  and  $\vdash_{NL_\circ} (\circ\varphi \wedge \circ\psi) \rightarrow \circ(\varphi\#\psi)$  for each binary  $\# \in X$  (Prop\*).

The first condition is obviously satisfied since  $\circ\bar{0}$  is an axiom of the logic. Next proposition shows that, in fact, the other conditions are also satisfied.

**Proposition 1**  $NL_\circ$  satisfies (Prop\*).

In the context of LFIs, it is a desirable property to recover the classical reasoning by means of the consistency connective  $\circ$  (see [3]). Specifically, let **CPL** be classical propositional logic. If  $L$  is a given LFI such that its reduct to the language of **CPL** is a sublogic of **CPL**, then a DAT (Derivability Adjustment Theorem) for  $L$  with respect to

**CPL** is as follows: for every finite set of formulas  $\Gamma \cup \{\varphi\}$  in the language of **CPL**, there exists a finite set of formulas  $\Theta$  in the language of **L**, whose variables occur in formulas of  $\Gamma \cup \{\varphi\}$ , such that:

$$\text{(DAT)} \quad \Gamma \vdash_{\mathbf{CPL}} \varphi \text{ iff } \circ(\Theta) \cup \Gamma \vdash_{\mathbf{L}} \varphi.$$

When the operator  $\circ$  enjoys the propagation property in the logic **L** with respect to the classical connectives then the DAT takes the following, simplified form: for every finite set of formulas  $\Gamma \cup \{\varphi\}$  in the language of **CPL**,

$$\text{(PDAT)} \quad \Gamma \vdash_{\mathbf{CPL}} \varphi \text{ iff } \{\circ p_1, \dots, \circ p_m\} \cup \Gamma \vdash_{\mathbf{L}} \varphi$$

where  $\{p_1, \dots, p_m\}$  is the set of propositional variables occurring in  $\Gamma \cup \{\varphi\}$ .

The fact that  $\circ$  has the propagation property it allows us to prove the following PDAT theorem for  $\text{NL}_{\circ}^{\leq}$ .

**Theorem 5.5 (PDAT for  $\text{NL}_{\circ}^{\leq}$ )** *Let  $\Gamma \cup \{\varphi\}$  be a finite set of formulas in the language of **CPL** and let  $\{p_1, \dots, p_m\}$  the set of propositional variables appearing in  $\Gamma \cup \{\varphi\}$ . Then*

$$\Gamma \vdash_{\mathbf{CPL}} \varphi \text{ iff } \Gamma \vdash_{\text{NL}_{\circ}^{\leq}} \left( \left( \bigwedge_{i=1}^n \circ(p_i) \right) * \left( \bigwedge_{i=1}^n \circ(p_i) \right) \right) \rightarrow \varphi.$$

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