Proof systems for Geometric theories (PROGEO)

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Abstract. *We plan to study the problem of finding conservative extensions of first order logics. In this project we intend to establish a systematic procedure for adding geometric theories in both intuitionistic and classical logics, as well as to extend this procedure to bipolar axioms, a generalization of the set of geometric axioms. This way, we obtain proof systems for several mathematical theories, such as lattices, algebra and projective geometry, being able to reason about such theories using automated deduction.*

Resumo. *Neste projeto de pesquisa, pretendemos estudar o problema de estender, conservativamente, lógicas de primeira-ordem. Pretendemos estabelecer um procedimento sistemático para adicionar teorias geométricas e extensões em lógicas clássica e intuicionista. Dessa forma, obtemos sistemas de provas para diversas teorias matemáticas, tais como reticulados, álgebra e geometria projetiva, provendo sistemas automáticos de dedução para tais teorias.*

1. Introduction

One of the advantages of using sequent systems as a frameworks for logical reasoning is that the resulting calculi are often simple, have good proof theoretical properties (like cut-elimination, consistency, etc) and can be easily implemented, e.g., using rewriting.

Hence it would be heaven if we could add axioms in mathematical theories to first order logics and reason about them using all the machinery already built for the sequent framework. Indeed, the general problem of extending standard proof-theoretical results obtained for pure logic to certain class of non-logical axioms has been focus of attention for quite some time now.

The main obstacle for this agenda is that adding non-logical axioms to systems while still maintaining the good proof theoretical properties it is not an easy task. In fact, as described in [Negri and von Plato 1998], if A, B are atoms and the axioms $\vdash A \supset B$ and $\vdash A$ are added to the sequent system LJ for intuitionistic logic [Gentzen 1935a], then the sequent $\vdash B$ can be derived using *cut*:

$$\dfrac{\dfrac{}{\vdash A} \qquad \dfrac{\dfrac{}{\vdash A \supset B} \qquad \dfrac{\dfrac{}{A \vdash A}\,init \qquad \dfrac{}{B \supset B}\,init}{A, A \supset B \vdash B}\supset L}{A \vdash B}\,cut}{\vdash B}\,cut$$

But it is easy to see that there is no proof of this sequent *without* cut. That is, the resulting system is not *cut-free*: applications of the rule *cut* can not be eliminated.

One way of circumventing this problem is by treating axioms as *theories*, added to the sequent context. This is already in Gentzen's consistency proof of elementary arithmetic in [Gentzen 1935b]. Now the derivations have only logical axioms as premisses, and cut elimination applies. In the example above, we can derive B from $A, A \supset B$ without a problem

$$\dfrac{\dfrac{}{A \vdash A}\ init \qquad \dfrac{}{A, B \vdash B}\ init}{A, A \supset B \vdash B}\ \supset L$$

But we can do better by transforming the axioms above into *inference rules*. In fact, if A, B are atomic formulas and C an arbitrary formula then, in the presence of $A \supset B$, if B proves C then A also proves C. On the other hand, in the presence of A, if A proves C, then C is provable (the A is irrelevant since it is *already there*). This induces the inference rules

$$\dfrac{\Gamma, B \vdash C}{\Gamma, A \vdash C}\ A \supset B \qquad \dfrac{\Gamma, A \vdash C}{\Gamma \vdash C}\ A$$

The sequent $\vdash B$ now has the (cut-fre) proof

$$\dfrac{\dfrac{\dfrac{}{B \vdash B}\ init}{A \vdash B}\ A \supset B}{\vdash B}\ A$$

In this project, we intend to propose a systematic way of adding inference rules to sequent systems. The proposal will be based on the notions of *focusing* and *polarities*, illustrated next.

2. A gentle introduction to polarities and focusing

We will start by generalizing the example above. Let B be a formula and Γ be a multiset of formulas. Consider attempting to build a proof of the following two-sided sequent

$$\Gamma, A_1 \supset \cdots \supset A_n \supset A_0 \vdash B,$$

in which the distinguished implication is such that $n \geq 1$ and $A_0, \ldots, A_n$ are atomic formulas. In general, there are many ways to proceed with attempting to build a cut-free proof of this sequent and we characterize them as one of the following four possibilities. This sequent can be the conclusion of

1. a structural rule (weakening or contraction) or the initial rule;
2. a right introduction rule, if B is not an atomic formula;
3. a left-introduction rule that introduces a formula in Γ; or
4. the implication-introduction rule that introduces the distinguished implication.

The number of possible choices here could be large, particularly if Γ contains a large number of formulas. If we chose the fourth of these possibilities, the proof would look as follows (at least in the intuitionistic setting):

$$\dfrac{\Gamma \vdash A_1 \qquad \Gamma, A_2 \supset \cdots \supset A_n \supset A_0 \vdash B}{\Gamma, A_1 \supset \cdots \supset A_n \supset A_0 \vdash B}\ L\supset$$

Note that we again have a large number of possible ways to proceed in attempting to prove the right premise: indeed, if $n \geq 2$, we have all the same choices as before. Clearly, those choices—and their multiplicative effects as we search for a sequence of inference steps that terminates in a proof—are in desperate need of being structured somehow. Focused proof systems provide such structure using the following two devices.

Focused rule application If you chose to apply the implication-left introduction on the distinguished implication, then you also commit to repeat the implication-left rule on the right premise until the atomic formula A_0 results. That is, the left-introduction applied to the distinguished implication results in the following derived inference rule

$$\frac{\Gamma \vdash A_1 \quad \dots \quad \Gamma \vdash A_n \qquad \Gamma, A_0 \vdash B}{\Gamma, A_1 \supset \cdots \supset A_n \supset A_0 \vdash B}\ L{\supset} \quad n \text{ times.}$$

Polarization Although the focused application of inference rules provides structure to attempts to build proofs, there are still so many remaining choices, that it is possible to impose two different "protocols" for restricting choices further. The Q-protocol insists that the first n premises above are trivial, meaning that they are proved by the initial rule. Following that protocol, we have $A_i \in \Gamma$ for $1 \leq i \leq n$. Thus, if we set Γ' to be the result of removing all occurrences of $A_1, \dots, A_n$ from Γ, then the derived inference rule above becomes

$$\frac{\Gamma', A_1, \dots, A_n, A_0 \vdash B}{\Gamma', A_1, \dots, A_n, A_1 \supset \cdots \supset A_n \supset A_0 \vdash B}\ .$$

The second protocol, the T-protocol insists that the right-most premise is trivial: that is, A_0 and B are the same atomic formula. Thus, the derived inference rule above becomes

$$\frac{\Gamma \vdash A_1 \quad \dots \quad \Gamma \vdash A_n}{\Gamma, A_1 \supset \cdots \supset A_n \supset A_0 \vdash A_0}\ .$$

Using the Q-protocol, the proof-search semantics of the implication $A_1 \supset \cdots \supset A_n \supset A_0$ is given by *forward-chaining*: if you have assumptions $A_1, \dots, A_n$ then you can add the assumption A_0. Using the T-protocol, the proof-search semantics of the same implication is given by *back-chaining*: in order to prove the conclusion A_0, attempt instead to prove each of $A_1, \dots, A_n$. The names for the Q and T protocols comes from Danos, Joinet, and Schellinx [Danos et al. 1995]: in the Q protocol, the tail ("queue") of an implication yields a trivial premise while in the T protocol, the head ("tête") of an implication yields a trivial premise.

A more modern and flexible presentation of the Q and T protocols speaks, instead, of the *polarity* of formulas: for this example, the polarity given to atomic formulas is the most relevant. In particular, if all atomic formulas have a positive polarity, the Q-protocol is enforced, while if all atomic formulas have a negative polarity, the T-protocol is enforced.

The base systems we will consider in this project are the focused proof systems for classical and intuitionistic logics, *LKF* and *LJF*, respectively [Liang and Miller 2007, Liang and Miller 2009]. Those systems extend both the notion of focusing and polarity to all formulas, moving beyond the example above involving only implications and atomic formulas. In particular, *focused rule applications* imply that focus is transferred from conclusion to premises in derivations. This process goes on until either the focused phase ends (depending on the *polarity* of the focused formula), or the derivation ends. Once the focus is released, the formula is eagerly decomposed into subformulas, which are ultimately *stored* in the context.

Reading derivations from the root upwards, this forces a sequent derivation to be organized into *focused phases*, each of them corresponding to an application of a *synthetic inference rule* [Chaudhuri 2008], where the focused formula is rewritten into (some of) its subformulas.

There is a class of formulas corresponding to particularly interesting synthetic rules: the *bipolars*. Bipolars are formulas in which polarity can change at most once among its subformulas. This means that focusing on a bipolar A gives rise to (possibly many) synthetic inference rules having simple shape, with leaves involving only atomic subformulas of A. We call a synthetic inference rule corresponding to the bipolar A a *bipole* for A.

In this project, we will present a careful study of bipoles, giving a fresh view to an old problem: how to incorporate inference rules encoding axioms into proof systems for classical and intuitionistic logics.

A key step in transforming a formula into synthetic inference rules involves attaching a polarity to atomic formulas and to some logical connectives. Since there are different choices for assigning polarities, it is possible to produce different synthetic inference rules for the same (unpolarized) formula. In the example above, there are (at most) 2^{n+1} different possible polarizations for the atomic formulas in $A_1 \supset \cdots \supset A_n \supset A_0$, each of them corresponding to a different bipole.

We show that this flexibility allows for the generalization of different approaches for transforming axioms into sequent rules present in the literature (more notably the series of works [Negri 2003, Negri and von Plato 2011, Negri 2016] and [Viganò 2000]).

3. A case study: geometric axioms

The main challenge in this effort is to determine a general procedure that guarantees that such extensions preserve good proof-theoretical properties.

A remarkable step in that direction was the careful investigation of *geometric axioms*. Geometric axioms are first-order formulas that can be converted into (natural deduction/sequent) inference rules having "a certain simple form in which only atomic formulas play a critical part", as described by Simpson [Simpson 1994]. And this "simple rules for atomic formulas" motto seems to be the core of success in this endurance in the approaches/extensions present in the literature [Dyckhoff and Negri 2015]. In this work, we come back to the inception of the axioms-as-rules problem, showing that the combination of bipolars and focusing is the real essence of "simple rules for atomic formulas".

There are many examples of geometric theories in different areas of logic and mathematics, such as geometry, algebra, topology and category theory (see some examples in Section 5).

We will illustrate next how to translate this class of axioms into synthetic inference rules.

Definition 1 *A* geometric implication *is a first-order formula having the form*

$$\forall \overline{z}(P_1 \wedge \ldots \wedge P_m \supset \exists \overline{x}_1 M_1 \vee \ldots \vee \exists \overline{x}_n M_n),$$

where each P_i is an atomic formula, each M_j is a conjunction of atomic formulas $Q_{j_1}, \ldots, Q_{j_{k_j}}$, and none of the variables in the lists $\overline{x}_1, \ldots, \overline{x}_n$ are free in P_i. A geometric theory *is a finite set of geometric implications. We shall also assume that if the list of variables $\bar{x}_i$ is empty then M_i is just an atom: otherwise, this formula can be written as a conjunction of geometric implications.*

An example of a geometric implication is the *transitivity* axiom, stating that, for a binary relation R, if x is related to y and y is related to z then x is related to z

$$4 = \forall x, y, z.(R(x, y) \wedge R(y, z)) \supset R(x, z)$$

Now, for *polarizing* this formula in *LKF* or *LJF*, we can give to the atomic formula R and the conjunction a *positive* or a *negative* polarity (the quantifiers and implication are *neutral* in *LKF*/*LJF*). We then obtain the following four poralized formulas (bipolars)

$$\forall x, y, z.(R(x, y)^{\pm} \wedge^{\pm} R(y, z)^{\pm}) \supset R(x, z)^{\pm}$$

As one can expect, different polarizations can give rise to different bipoles (inference rules). For this example, focusing on each and all these formulas in *LJF* (it holds also for *LKF*) will produce the following two inference rules (bipoles)[1]

$$\frac{R(x, z), \Gamma \vdash C}{R(x, y), R(y, z), \Gamma \vdash C}\ 4_{GRS} \qquad \frac{\Gamma \vdash R(x, y) \quad \Gamma \vdash R(y, z)}{\Gamma \vdash R(x, z)}\ 4_{RR}$$

The rule 4_{GRS} appears in [Negri 2005] and corresponds to backward-chaining, while the rule 4_{RR} is the transitivity rule studied in [Viganò 2000], corresponding to forward-chaining. This implies that these works are different faces of the same coin, the latter being minted from focusing and polarization.

Moreover, we address these issues with a uniform presentation in both classical and intuitionistic first-order logics.

4. Beyond geometric axioms

It turns out that the set of bipolar formulas is strictly greater than the set of geometric formulas. As an example, in set theory, the following implication relates the subset and membership predicates

$$\forall yz.(\forall x(x \in y \supset x \in z) \supset y \subseteq z).$$

[1]For details about the systems *LKF* and *LJF*, as well as the process of transforming axioms to synthetic rules using focusing, please refer to [Marin et al. 2020].

This formula yields a bipolar (but not geometric) formula in both *LKF* and *LJF* under any polarization of the binary atomic predicates $\in$ and $\subseteq$. Assuming that these predicates are given positive polarity, the corresponding *LJF*-synthetic inference rule is

$$\frac{x \in y, \Gamma \vdash x \in z \qquad y \subseteq z, \Gamma \vdash E}{\Gamma \vdash E} .$$

Assuming that these predicates are given negative polarity, the corresponding *LJF*-synthetic inference rule is

$$\frac{x \in y, \Gamma \vdash x \in z}{\Gamma \vdash y \subseteq z} .$$

In both of these synthetic inference rules, x is an eigenvariable for that rule.

This means that our bipoles/focusing method generalizes and goes beyond the ones present in the literature. In fact, it classifies all and only axioms that can be transformed to sequent rules.

The sole responsible for this is the fact that, in bipolars, the polarity can change at most once among its subformulas. This means that focusing on a bipolar will completely decompose such formula until getting to atoms, which will be either stored in the context (in this case it appears in premises), or will be principal in the initial axiom (in which case it appears in the conclusion). In this sense, the rules corresponding to the bipolars – the bipoles – can be seen as introduction rules for *atoms*.

5. Examples in Mathematics

We finish this study proposal by enumerating some examples of axioms in mathematics that can be analyzed within this project. Some of them appear in the book [Negri and von Plato 2011], some are unpublished.

1. *Partial order.* Assuming the domain $\mathcal{D} = \{a, b, c \ldots\}$ and a binary relation $\leq$ in $\mathcal{D}$, we say that $\leq$ is a partial order over $\mathcal{D}$ if:
   PO1 $\forall a. a \leq a$ (reflexivity).
   PO2 $\forall a, b. (a \leq b) \wedge (b \leq a) \supset a = b$ (anti-simety).
   PO3 $\forall a, b. (a \leq b) \wedge (b \leq c) \supset a \leq c$ (transitivity).
2. *Strict partial order.* Assuming the domain $\mathcal{D} = \{a, b, c \ldots\}$ and a binary relation $<$ in $\mathcal{D}$, we say that $<$ is a strict partial order over $\mathcal{D}$ if it satisfies transitivity and:
   PO4 $\forall a. \neg(a < a)$ (irreflexivity).
   It is easy to see that, if a relation $<$ satisfies $[PO3]$ and $[PO4]$, then it also satisfies:
   PO5 $\forall a, b. (a < b) \supset \neg(b < a)$.
3. *Projective geometry.* An axiomatization of projective geometry starts by defining the basic domain and relations. Denoting points by $a, b, c, \ldots$ and lines by $l, m, n, \ldots$, we have that the basic relations are $a = b$, $l = m$ and $a \in l$. We will consider then the reflexivity and transitivity described before together with:
   ER1 $\forall a, b. (a = b) \supset (b = a)$ (simmetry).

4. *Normal modal logic.* Extensions of modal logic K are determined by adding relational axioms to the original system. We describe some bellow.
   E $\forall w, o, r.(wRo \wedge wRr) \supset oRr$ (euclideanity)
   D $\forall w.\exists o.wRo$ (seriality)
5. *Torsion abelian groups.* The main axiom of torsion abelian groups, that says that all objects has finite order, can be described as
$$\forall x.\top \supset \bigvee_{n=1}^{\infty} nx = 0$$
6. *Local rings.* The main axiom of torsion abelian groups, that says that there exists exactly one maximal ideal, can be described as
$$\forall x.\top \supset (\exists y.(xy = 1)) \vee (\exists y.(1 - x)y = 1))$$
7. *Set theory.* There are many axioms in set theory falling into the bipolar setting. As an example, the following implication relates the subset and membership predicates:
$$\forall yz.(\forall x(x \in y \supset x \in z) \supset y \subseteq z).$$

6. Conclusion

We have illustrated how the notion of synthetic inference rule that is provided by sequent calculus notions of polarization and focusing can be used to provide inference rules that capture certain classes of axioms.

In particular, focused proof systems naturally lead to the notion of bipolar formulas and these result in synthetic inference rules that only need to mention atomic formulas.

We show that geometric formulas are examples of such bipolar formulas and that polarized versions of such formulas yield known inference systems derived from geometric formulas. Certain subsets of geometric formulas admit more than one polarization and these variations explain the forward-chaining and backward-chaining variants of their synthetic inference rules. Additionally, all of these results work equally well in both classical and intuitionistic logics using the corresponding *LKF* and *LJF* focused proof systems.

With this project, we plan to develop the application of such a framework of focusing and bipoles to mathematical theories.

This project will be in collaboration with:

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