# Another Calculational Proof of Cantor's Theorem

**Márcia R. Cerioli**[1], **Renata de Freitas**[2], **Petrucio Viana**[2]

[1]Instituto de Matemática e
Programa de Engenharia de Sistemas e Computação – COPPE
Universidade Federal do Rio de Janeiro (UFRJ) – Rio de Janeiro – RJ – Brazil

[2]Instituto de Matemática e Estatística
Universidade Federal Fluminense (UFF) – Niterói – RJ – Brazil

marcia@cos.ufrj.br, {renatafreitas,petrucio_viana}@id.uff.br

***Abstract.*** *E. Dijkstra and J. Misra* [American Mathematical Monthly, 108:440–443 (2001)] *presented a calculational proof of Cantor's Theorem. Their proof is based essentially on the Axiom of Choice. In this note, we present another calculational proof which does not appeal, at least directly, to the Axiom of Choice. Our proof is based only on logical steps and a heuristic guidance analogous to the one used by Dijkstra and Misra in their proof.*

## 1. Introduction

In [Dijkstra and Misra 2001], E. Dijkstra and J. Misra presented a calculational proof—based on a heuristic guidance provided by the proof design—of Cantor's Theorem, that there is no 1-1 correspondence between a set $S$ and its power set $\mathcal{P}(S)$. The general strategy of their proof is to show that there is no surjective function $F : S \to \mathcal{P}(S)$. To this, they prove in an ingenious way that no function $F : S \to \mathcal{P}(S)$ has a right inverse $g : \mathcal{P}(S) \to S$. Therefore, their proof is based essentially on the Axiom of Choice [Bernays 1941].

In this note, we present another calculational proof based only on logical equivalences—and a heuristic guidance analogous to that discovered by Dijkstra and Misra—which does not appeal, at least directly, to the Axiom of Choice. Although we refer to sets, relations and functions, we work in the equational environment of allegories [Freyd and Scedrov 1990], providing "pointless" proofs of some known results. As a subproduct, our proofs show that (some of) these results hold in more general environments than that of the allegory of sets, relations and functions.

## 2. Review on functions

In this section, we review some basic useful concepts and results about relations and functions.

Let $X$ and $Y$ be sets. The *universal relation* from $X$ to $Y$ is the set $X \times Y$. In general, a *relation* from $X$ to $Y$ is a subset $R$ of $X \times Y$. In particular, the *empty relation* from $X$ to $Y$ is the relation $\emptyset = \{(x, y) \in X \times Y : x \neq x \wedge y \neq y\}$ and the *identity relation* from $X$ to $X$ is the relation $\mathsf{Id}_X = \{(x, y) \in X \times X : x = y\}$. Given $x \in X$ and $y \in Y$, as usual, we write "$xRy$" instead of "$(x, y) \in R$."

Being sets, relations are subject to the usual operations of $\cup$ (union), $\cap$ (intersection) and $^-$ (complementation with respect to $X \times Y$) and for all sets $X$ and $Y$, the set

of all relations from $X$ and $Y$ endowed with $\cup$, $\cap$, $^-$, $\emptyset$ and $X \times Y$ is an atomic and complete Boolean algebra. Besides, being sets *of ordered pairs*, relations are also subject to the usual operations of $\circ$ (composition), and $^{-1}$ (reversion), and the class of all relations from $X$ to $X$ endowed with $\cap$, $\circ$, $^{-1}$, and $\mathsf{Id}_X$ is an allegory. Accordingly, we apply the usual arithmetical Boolean and allegoric properties of relations without further notice.

Let $R$ be a relation from $X$ to $Y$. We say that $R$ is:

(1) *total* if for every $x \in X$ there exists $y \in Y$ such that $xRy$ or, in "pointless" notation, $\mathsf{Id}_X \subseteq R \circ R^{-1}$;
(2) *functional* if for all $x \in X$ and $y_1, y_2 \in Y$, $xRy_1$ and $xRy_2$ implies $y_1 = y_2$ or, in "pointless" notation, $R^{-1} \circ R \subseteq \mathsf{Id}_Y$;
(3) *injective* if for all $x_1, x_2 \in X$ and $y \in Y$, $x_1Ry$ and $x_2Ry$ implies $x_1 = x_2$ or, in "pointless" notation, $R \circ R^{-1} \subseteq \mathsf{Id}_X$;
(4) *surjective* if for every $y \in Y$ there exists $x \in X$ such that $xRy$ or, in "pointless" notation, $\mathsf{Id}_Y \subseteq R^{-1} \circ R$;
(5) *function* if it is total and functional;
(6) *active* if it is injective and surjective;
(7) *bijection* if it is an active function.

In what follows, when convenient, if $R$ is a function we denote $R$ by $f$, write "$f : X \to Y$" instead of "$R \subseteq X \times Y$", and write "$f(x) = y$" instead of "$xRy$" for $x \in X$ and $y \in Y$. Besides, if $g : Y \to Z$ is also a function, we write "$g(f(x))$" to denote the unique element $z \in Z$ such that $g(f(x)) = z$.

Proposition 1 provides characterizations of injectivity and surjectivity in terms of the existence of the so called *left* end *right* inverses.

**Proposition 1** For all sets $X$ and $Y$, and function $f : X \to Y$, we have:

*(a)* $f$ is injective iff there exists a functional relation $R \subseteq Y \times X$ such that $f \circ R = \mathsf{Id}_X$, i.e., for all $x_1, x_2 \in X$ and $y \in Y$, if $x_1fy$ and $yRx_2$, then $x_1 = x_2$.

*(b)* $f$ is surjective iff there exists a function $g : Y \to X$ such that $g \circ f = \mathsf{Id}_Y$, i.e., $f(g(y)) = y$ for every $y \in Y$.

PROOF. (a) ($\Rightarrow$) Take $R = f^{-1} \subseteq Y \times X$. To prove $R$ is functional, notice that $R^{-1} \circ R = (f^{-1})^{-1} \circ f^{-1} = f \circ f^{-1} \subseteq \mathsf{Id}_X$, since $f$ is injective. To prove $R$ is a right inverse, notice that, since $f$ is total, $\mathsf{Id}_X \subseteq f \circ f^{-1} = f \circ R$.

($\Leftarrow$) First we prove that $f \subseteq R^{-1}$. In fact:

$$\begin{array}{rcl} f & = & \mathsf{Id}_X \circ f \\ & = & (\mathsf{Id}_X)^{-1} \circ f \\ & = & (f \circ R)^{-1} \circ f \text{ (since } R \text{ is right inverse)} \\ & = & R^{-1} \circ f^{-1} \circ f \\ & \subseteq & R^{-1} \circ \mathsf{Id}_Y \text{ (since } f \text{ is functional)} \\ & = & R^{-1} \end{array}$$

Now to prove $f$ is injective, we have $f \circ f^{-1} \subseteq R^{-1} \circ (R^{-1})^{-1} = R^{-1} \circ R \subseteq \mathsf{Id}_X$, since $R$ is functional.

(b) ($\Rightarrow$) Since $f^{-1}$ is a relation, by the Axiom of Choice, take a functional relation $g \subseteq f^{-1}$ such that $\mathsf{Id}_Y \cap (g \circ g^{-1}) = \mathsf{Id}_Y \cap (f^{-1} \circ (f^{-1})^{-1})$. (This last equality express in the language of allegories that $g$ and $f^{-1}$ have the same domain.) To prove $g$ is total, notice that $\mathsf{Id}_Y = \mathsf{Id}_Y \cap \mathsf{Id}_Y \subseteq$ (since $f$ is surjective) $\mathsf{Id}_Y \cap (f^{-1} \circ f) = \mathsf{Id}_Y \cap (f^{-1} \circ (f^{-1})^{-1}) =$ (since $g$ and $f^{-1}$ have the same domain) $\mathsf{Id}_Y \cap (g \circ g^{-1}) \subseteq g \circ g^{-1}$. To prove $g$ is left inverse, notice that $g \circ f \subseteq f^{-1} \circ f \subseteq \mathsf{Id}_Y$, since $f$ is functional.

($\Leftarrow$) First we prove that $g \subseteq f^{-1}$. In fact:

$$\begin{array}{rcl} g & = & g \circ Id_X \\ & \subseteq & (g \circ f) \circ f^{-1} \text{ (since } f \text{ is total)} \\ & \subseteq & \mathsf{Id}_Y \circ f^{-1} \text{ (since } g \text{ is left inverse)} \\ & = & f^{-1} \end{array}$$

Now to prove $f$ is surjective, we have $\mathsf{Id}_Y = g \circ f \subseteq f^{-1} \circ f$, since $g$ is left inverse. ■

Observe that in Proposition 1(a), the domain of $R$ may be just a proper subset of $X$, whereas in Proposition 1(b) it must be the whole of $Y$. On the other hand, if we are dealing with sets, relations and functions, Proposition 1(a) can be improved to guarantee that $R$ is total. In fact, the usual reasoning, based on points, goes by considering two cases. If $X = \emptyset$, we have $f = \emptyset$ and the equivalence is vacuously true. If $X \neq \emptyset$, let $a \in X$ and define $R \subseteq Y \times X$ by setting, for all $y \in Y$, either $yRx$ when there exists $x \in X$ such that $f(x) = y$ or $yRa$ otherwise. To prove $R$ is total, let $y \in Y$. We have two cases. If there exists $x \in X$ such that $f(x) = y$, then by definition of $R$, we have $yRx$. Otherwise, by definition of $R$, we have $yRa$. The calculational proof of Cantor's Theorem presented in Section 5 is based on the more general "pointless" Proposition 1(a).

The proof of Proposition 1(b) uses a version of the Axiom of Choice, due to P. Bernays, reformulated adequately in the language of allegories. Bernays version warrants that

> for every function $F$ there exists an inverse function, i.e., a function which is a subclass of the converse class of $F$ and whose domain is the converse domain [i.e., the range] of $F$.
> Thus the Axiom IV [i.e. the Axiom of Choice] is equivalent in content to the assumption that for every function there exists an inverse function ([Bernays 1941], pp. 1–2).

In other words, if we are dealing with sets, relations and functions, this version is equivalent to the usual Axiom of Choice, i.e, (in Bernays words): "For every class $C$ of non-empty sets there exists a function, having $C$ as its domain, which assigns to every set belonging to $C$ one of its elements." It follows that any proof involving sets, relations and functions based on Proposition 1(b) depends essentially on the Axiom of Choice and does not transfer the result to certain forms of set theory.

## 3. Cantor's Theorem

In this section, we state Cantor's Theorem [Cantor 1891] and present the historical and usual proofs of it.

Using the current set theoretical language but retaining some of the original wording, Cantor's statement of the theorem is as follows:

> For any given set $L$ another set $M$ can be put on the side, which is of greater cardinality than $L$.
>
> Nowadays, we state the theorem as:

**Theorem 1 [Cantor 1891]** *For every set $X$, $|X| < |\mathcal{P}(X)|$.*

Cantor's original proof takes the real interval $[0,1]$ as $L$ and put the set $\{f \mid f : [0,1] \to \{0,1\}\}$ of all functions from $[0,1]$ to $\{0,1\}$ as $M$. The nature of $[0,1]$ plays no role in his proof and, since we are talking about sets and functions, the sets $\{f \mid f : [0,1] \to \{0,1\}\}$ and $\mathcal{P}(L)$ have the same cardinality.

Cantor's original reasoning has two parts. First he proves the cardinality of $M$ is not greater than the cardinality of $L$, or better, that the cardinality of $L$ is less or equal than the cardinality of $M$.

> The fact that $M$ has no smaller [cardinality] than $L$, follows from the fact that subsets of $M$ can be specified, which have the same power as $L$, for example, the subset consisting of all the functions of $x$ that have the value $1$ for a single value $x_0$ of $x$, and $0$ for all other values of $x$.
>
> Nowadays, this part of the proof is encompassed in the following lemma:

**Lemma 1** *For every set $X$ there is an injective function from $X$ to $\mathcal{P}(X)$.*

PROOF. Let $f : X \to \mathcal{P}(X)$, defined by $f(x) = \{x\}$, for every $x \in X$. The Axiom of Pair warrants that $f$ is total, whereas the Axiom of Extensionality warrants that $f$ is functional. To prove that $f$ is injective, let $x, y \in X$ and suppose that $f(x) = f(y)$. Hence, $\{x\} = \{y\}$, and so, again by the Axiom of Pair $x = y$. ■

Afterwards, Cantor continues the proof, using an alleged bijection between $L$ and $M$ to define a two-variable function from $L \times L$ to $M$ and a function belonging to $M$ not in the image of the bijection. Cantor's argument can be put in the following form:

> $M = \{f \mid f : [0,1] \to \{0,1\}\}$ does not have the same cardinality as $L = [0,1]$, because otherwise there exists a bijective function $\alpha : L \to M$ and a function $\phi : L \times L \to M$ defined by setting $\phi(x,y) = \alpha(x)(y)$, for all $x, y \in L$. Now, define $g : L \to \{0,1\}$ by setting $g(x) = 1 - \phi(x,x)$, for every $x \in L$. By definition of $g$, we have both $g \in M$ and
>
> $$g(x) \neq \phi(x,x), \text{ for every } x \in L. \tag{1}$$
>
> Now, since $\alpha$ is bijective, there exists $l \in L$ such that $\alpha(l) = g$. So, $\alpha(l)(x) = g(x)$, for every $x \in L$. Hence, by definition of $\phi$, $\phi(l,l) = \alpha(l)(l) = g(l)$, contradicting (1).

Nowadays, this part of the result is presented according to E. Zermelo's approach ( [Zermelo 2010], page 167) which implemented three small changes in Cantor's argument. First, he considered a generic set $X$ instead of the particular $[0,1]$ (as we mentioned, this change is not essential). Second, he replaced the set $\{f \mid f : [0,1] \to \{0,1\}\}$ of all characteristics functions of $X$ by the power set $\mathcal{P}(X)$ (again a non-essential change, since Cantor's proof is based only in the fact that $[0,1]$ is a set). Third, he replaced proving that there is no surjective function from $X$ to $\mathcal{P}(X)$ by proving the dual statement that there is no injective function from $\mathcal{P}(X)$ to $X$. For some historical reason yet to be unveiled, this latest modification has not entered the mainstream of mathematical texts and nowadays we state and prove the result in the following manner:

**Lemma 2** *For every set $X$ there is no surjective function from $X$ to $\mathcal{P}(X)$.*

PROOF. Suppose, for a contradiction, that $f : X \to \mathcal{P}(X)$ is surjective. Let $Y = \{y \in X \mid y \notin f(y)\}$. Since $f$ is surjective, there exists $x \in X$ such that $f(x) = Y$. By definition of $Y$, we have $x \in Y$ iff $x \notin f(x)$. Now, since $f(x) = Y$, we have $x \in f(x)$ iff $x \notin f(x)$, a contradiction. ■

The key ingredient of the proof of Lemma 2 is—as calculational provers usually say—the rabbit "$Y = \{y \in X \mid y \notin f(y)\}$" pulled out of the hat before the public's amazed eyes. Although this result is proven in the manner above in (almost all) books and articles in which Cantor's Theorem is mentioned, to the best of our knowledge there is no text (besides [Dijkstra and Misra 2001]) where the definition of this set is discussed or, at least, motivated. In the next section, we examine Dijkstra and Misra solution for this problem.

## 4. Calculational proof of Cantor's Theorem via Lemma 2

In this section, we examine some aspects of the Dijkstra and Misra's calculational proof of Cantor's Theorem via Lemma 2 [Dijkstra and Misra 2001], that has as purposes (1) "to show that formal arguments need not be lengthy at all"; and (2) "to show the strong heuristic guidance that is available to us when we design such calculational proofs in sufficiently small, explicit steps."

In order to reduce the role played by the (apparent) guess of the rabbit "$Y = \{y \in X \mid y \notin f(y)\}$" in the proof of Lemma 2, Dijkstra and Misra's [Dijkstra and Misra 2001] present a calculational proof endowed with a heuristics which reveals in which sleeve the magician had the rabbit hidden. Here is their proof—based on a notation that is slightly different from ours—where $x \in X$, $Y \in \mathcal{P}(X)$, $F : X \to \mathcal{P}(X)$ and $g : \mathcal{P}(X) \to X$. Comments follow.

$$
\begin{array}{ll}
 & \mathsf{Id}_{\mathcal{P}(X)} \neq F \circ g \\
\Leftrightarrow & \{\text{definition of } \mathsf{Id}_{\mathcal{P}(X)}, \neq\} \\
 & \langle \exists Y :: Y \neq (F \circ g)(Y) \rangle \\
\Leftrightarrow & \{\text{definition of } \circ\} \\
 & \langle \exists Y :: Y \neq F(g(Y)) \rangle \\
\Leftrightarrow & \{\text{definition of } \neq\} \\
 & \langle \exists Y :: \langle \exists x :: x \in Y \not\Leftrightarrow x \in F(g(Y)) \rangle \rangle \\
\Leftrightarrow & \{\alpha \not\Leftrightarrow \beta \text{ equivalent to } \alpha \Leftrightarrow \neg\beta\} \\
 & \langle \exists Y :: \langle \exists x :: x \in Y \Leftrightarrow x \notin F(g(Y)) \rangle \rangle \\
\Leftarrow & \{\alpha(g(Y)) \text{ implies } \exists x \alpha(x), \text{ and } \exists \text{ is monotonic}\} \\
 & \langle \exists Y :: g(Y) \in Y \Leftrightarrow g(Y) \notin F(g(Y)) \rangle \\
\Leftarrow & \{\forall x \alpha(x) \text{ implies } \alpha(g(Y)), \text{ and } \exists \text{ is monotonic}\} \\
 & \langle \exists Y :: \langle \forall x :: x \in Y \Leftrightarrow x \notin F(x) \rangle \rangle \\
\Leftrightarrow & \{\text{definition of } \{x \mid \alpha(x)\}\} \\
 & \langle \exists Y :: Y = \{x \mid x \notin F(x)\} \rangle \\
\Leftarrow & \{\text{instantiation } Y := \{x \mid x \notin F(x)\}\} \\
 & \{x \mid x \neq F(x)\} = \{x \mid x \notin F(x)\} \\
\Leftrightarrow & \{= \text{ is reflexive}\} \\
 & true
\end{array}
$$

The proof starts with the statement that $F$ is not a left inverse of $g$, that is $g$ is not surjective. After a sequence of three set theoretical equivalences and one logical equivalence, it continues with a sequence of two logical implications, being the second one (from top to bottom) the first of the two ingenuous steps that reveal the place where the rabbit is hidden. The second one is the backward passage from $\langle \exists Y :: \langle \forall x :: x \in Y \Leftrightarrow x \notin F(x) \rangle \rangle$ to $\langle \exists Y :: Y = \{x \mid x \notin F(x)\} \rangle$, which concludes the work by bombastically revealing the rabbit in the magician's sleeve. The last two steps are an obvious instantiation followed by a logical equivalence.

In our opinion, there is a point where the Dijkstra and Misra's proof can be improved. Since their proof is based on the idea of proving that $g$ does not have a left inverse, as we discussed in sections 1 and 2, it is essentially based on the Axiom of Choice. In order to jump this hurdle, we ask if it is possible to present a calculational proof of Cantor's Theorem on Lemma 3 below, instead of Lemma 2.

**Lemma 3** *For every set $X$ there is no injective function from $\mathcal{P}(X)$ to $X$.*

Section 5 contains our solution for this problem.

## 5. Calculational proof of Cantor's Theorem via Lemma 3

In this section, we present a proof of Lemma 3. Except for a heuristic detour – elucidating the origin of a rabbit analogous to the infamous $Y = \{y \in X \mid y \notin f(y)\}$ which was employed in the usual proof of Lemma 2 — the proof comprises only a small sequence of logical and set theoretical steps.

First, observe that Lemma 3 can be stated as:

> For every set $X$ and function $f : \mathcal{P}(X) \to X$, we have that $f$ is not injective.

And that, according to Proposition 1(a), can be stated as:

> For every set $X$, function $f : \mathcal{P}(X) \to X$, and functional relation $g : X \to \mathcal{P}(X)$, there is a set $Z \in \mathcal{P}(X)$ such that $g(f(Z)) \neq Z$.

Now, we prove this last statement, applying logical steps as much as possible.

PROOF. We have to prove, for all $f : \mathcal{P}(X) \to X$ and $g : Y \to \mathcal{P}(X)$, that

$$\exists Z \in \mathcal{P}(X)[g(f(Z)) \neq Z].$$

But this is set theoretically equivalent to

$$\exists Z \in \mathcal{P}(X)\Big(\exists x \in X\Big(x \in g(f(Z)) \leftrightarrow x \notin Z\Big)\Big)$$

Now, observe that to prove this last statement, it suffices to exhibit an adequate $Z$ and an adequate $x$ for which $x \in g(f(Z))$ is equivalent to $x \notin Z$, because, in this way, we can start the proof with the tautology $x \notin Z \leftrightarrow x \notin Z$.

Now, it seems blatantly that a natural candidate for $Z$ may be the rabbit

$$\{y \in X \mid y \notin g(y)\}.$$

In fact, taking $Z = \{y \in X \mid y \notin g(y)\}$, and taking any element $x \in X$, we have:

$$x \in g(x) \Leftrightarrow x \notin Z.$$

Then, taking $f(Z) \in X$, we have:

$$f(Z) \in Z \Leftrightarrow f(Z) \notin g(f(Z))$$

and

$$f(Z) \notin Z \Leftrightarrow f(Z) \in g(f(Z)).$$

Besides:

$$\exists x \in X\Big([x \in g(x) \wedge x \in g(x)] \vee [x \notin g(x) \wedge x \notin g(x)]\Big)$$

is a tautology.

So, for the rabbit $Z$ defined as above, we have:

$$\exists x \in X\Big([x \in g(f(Z)) \wedge x \notin Z] \vee [x \notin g(f(Z)) \wedge x \in Z]\Big),$$

as required. ■

It follows a summary of our alternative calculational proof of Cantor's Theorem:

$$\begin{array}{ll}
 & \exists Z \in \mathcal{P}(X)\Big(g\big(f(Z)\big) \neq Z\Big) \\
\Leftrightarrow & \Big\{\text{set theoretical equivalence}\Big\} \\
 & \exists Z \in \mathcal{P}(X)\Big(\exists x \in X\Big(\big[x \in g\big(f(Z)\big) \wedge x \notin Z\big] \vee \big[x \notin g\big(f(Z)\big) \wedge x \in Z\big]\Big)\Big) \\
\Leftarrow & \Big\{\alpha\big[\{y \in X \mid y \notin g(y)\}\big] \text{ implies } \exists Z \alpha\big[Z\big]\Big\} \\
 & \exists x \in X\Big(\big[x \in g\big(f\big(\{y \in X \mid y \notin g(y)\}\big)\big) \wedge x \notin \{y \in X \mid y \notin g(y)\}\big] \vee \\
 & \big[x \notin g\big(f\big(\{y \in X \mid y \notin g(y)\}\big)\big) \wedge x \in \{y \in X \mid y \notin g(y)\}\big]\Big) \\
\Leftarrow & \Big\{\alpha\big[f\big(\{y \in X \mid y \notin g(y)\}\big)\big] \text{ implies } \exists x \alpha\big[x\big]\Big\} \\
 & \big[f\big(\{y \in X \mid y \notin g(y)\}\big) \in g\big(f\big(\{y \in X \mid y \notin g(y)\}\big)\big) \wedge \\
 & f\big(\{y \in X \mid y \notin g(y)\}\big) \notin \{y \in X \mid y \notin g(y)\}\big] \vee \\
 & \big[f\big(\{y \in X \mid y \notin g(y)\}\big) \notin g\big(f\big(\{y \in X \mid y \notin g(y)\}\big)\big) \wedge \\
 & f\big(\{y \in X \mid y \notin g(y)\}\big) \in \{y \in X \mid y \notin g(y)\}\big] \\
\Leftarrow & \Big\{\forall x \alpha\big[x\big] \text{ implies } \alpha\big[f\big(\{y \in X \mid y \notin g(y)\}\big)\big]\Big\} \\
 & \forall x \in X\Big(\big[x \in g(x) \wedge x \notin \{y \in X \mid y \notin g(y)\}\big] \vee \\
 & \big[x \notin g(x) \wedge x \in \{y \in X \mid y \notin g(y)\}\big]\Big) \\
\Leftrightarrow & \Big\{\text{definition of } \{x \mid \alpha(x)\}\Big\} \\
 & \forall x \in X\Big(\big[x \in g(x) \wedge x \in g(x)\big] \vee \big[x \notin g(x) \wedge x \notin g(x)\big]\Big) \\
\Leftrightarrow & \Big\{\text{excluded middle}\Big\} \\
 & true
\end{array}$$

## 6. Perspectives

This research left at least two nuts to be cracked. First, notice that the very last step of our calculational proof is based on the Excluded Middle. It is known that at the *topoi* level, the Excluded Middle is equivalent to the Axiom of Choice [Diaconescu 1975]. So, it seems that when viwed at some level of abstraction, our proof may depend of the Axiom of Choice. We think this is a matter that needs to be investigated. Second, and still related to the first question, there is a proof of Cantor's Theorem which are conducted inside intuitionistic logic [Maguolo and Valentini 1996]—where the Excluded Middle is not acceptable. This proof is far from calculational. There is a proposal to adapt the calculational proof style to deal with intuitionistic proofs [Bohórquez 2008]. So, we think that an interesting question is to find an equational intuitionitsc proof of Cantor Theorem based on a heuristic guidance provided by the proof design.

## References


Bernays, P. (1941). A system of axiomatic set theory–part II. *The Journal of Symbolic Logic*, 6(1):1–17.

Bohórquez, J. A. (2008). Intuitionistic logic according to Dijkstra's calculus of equational deduction. *Notre Dame Journal of Formal Logic*, 49(4):361–384.

Cantor, G. (1891). Über eine elementare Frage der Mannigfaltigskeitslehre. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 1(1):75–78.

Diaconescu, R. (1975). Axiom of choice and complementation. *Proceedings of the American Mathematical Society*, 51(1):175–178.

Dijkstra, E. and Misra, J. (2001). Designing a calculational proof of Cantor's theorem. *The American Mathematical Monthly*, 108(5):555–566.

Freyd, P. and Scedrov, A. (1990). *Categories, Allegories*. Elsevier, 1st edition.

Maguolo, D. and Valentini, S. (1996). An intuitionistic version of Cantor's theorem. *Mathematical Logic Quarterly*, 42(1):446–448.

Zermelo, E. (2010). *Ernst Zermelo-Collected Works/Gesammelte Werke: Volume I/Band I-Set Theory, Miscellanea/Mengenlehre, Varia*. Springer, 1st edition.