Quasi-N4-lattices and their logic

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Abstract. *The variety of quasi-N4-lattices (*$\mathsf{QN4}$*) was recently introduced as a non-involutive generalization of N4-lattices (algebraic models of Nelson's paraconsistent logic). While research on these algebras is still at a preliminary stage, we know that* $\mathsf{QN4}$ *is an arithmetical variety which possesses a ternary as well as a quaternary deductive term, enjoys equationally definable principal congruences and the strong congruence extension property. We furthermore have recently introduced an algebraizable logic having* $\mathsf{QN4}$ *as its equivalent semantics. In this contribution we report on the results obtained so far on this class of algebras and on its logical counterpart.*

1. Introduction

Nelson's constructive logic with strong negation [Nelson 1949] and its non-explosive counterpart, *Nelson's paraconsistent logic* [Almukdad and Nelson 1984] are two prominent non-classical logics whose algebraic models (resp. *Nelson algebras* and *N4-lattices*) have been extensively investigated [Sendlewski 1990, Odintsov 2003, Odintsov 2004, Spinks and Veroff 2018]. A recent series of papers introduced and investigated the class of *quasi-Nelson algebras* (a subvariety of commutative integral bounded residuated lattices) as a non-involutive generalization of Nelson algebras [Rivieccio and Spinks 2021]; the corresponding logic was axiomatized in [Liang and Nascimento 2019]. A similar abstraction was applied to the class of N4-lattices in [Rivieccio 2022], introducing the class of *quasi-N4-lattices* (i.e. non-involutive N4-lattices, or non-integral quasi-Nelson algebras); the corresponding logic is defined and shown to be algebraizable in [Lima Neto et al. 2022]. While the papers [Rivieccio 2020, Rivieccio 2021, Rivieccio and Spinks 2021, Rivieccio and Jansana 2021] already constitute, in our opinion, sufficient evidence for motivating the intrinsic interest in quasi-Nelson algebras, it remains to be seen to which extent the structure theory of the latter can be extended to the setting of quasi-N4-lattices. In this contribution we report on our current knowledge about these algebras and indicate some topics that appear to deserve further investigation.

2. Quasi-N4-lattices

As mentioned earlier, the models of Nelson's paraconsistent logic (N4-lattices, $\mathsf{N4}$) are a class of lattices that further possess an "intuitionistic-like" implication ($\to$) and a "strong" negation ($\sim$) satisfying the De Morgan laws and the double negation identity ($\sim\sim x \approx x$). Among N4-lattices, the models of Nelson's (explosive) logic (Nelson algebras, N) are precisely those that satisfy the identity $x \to x \approx y \to y$. A class of non-necessarily involutive Nelson algebras has been recently introduced under the name of *quasi-Nelson algebras* (QN), and Nelson algebras are precisely the quasi-Nelson algebras that satisfy the double negation identity. By applying a similar abstraction process to N4-lattices, one obtains non-necessarily involutive N4-lattices or *quasi-N4-lattices* (QN4): among them, the quasi-Nelson algebras are precisely those that satisfy the identity $x \to x \approx y \to y$, and the N4-lattices are precisely those that satisfy the double negation identity. We thus have the following inclusions (all proper): $\mathsf{N} \subseteq \mathsf{N4} \subseteq \mathsf{QN4}$, $\mathsf{N} \subseteq \mathsf{QN} \subseteq \mathsf{QN4}$. It is clear that $\mathsf{N} = \mathsf{QN} \cap \mathsf{N4}$, and an argument can be made to show that the variety generated by the class $\mathsf{QN} \cup \mathsf{N4}$ is properly contained in QN4 (see [Rivieccio and Jansana 2021, Sect. 3.1] and [Rivieccio 2022, Example 2.6]); we do not currently have an equational presentation for this variety.

In this section we present two equivalent presentations for quasi-N4-lattices; a corresponding (algebraizable) logic $\mathbf{L}_{\mathrm{QN4}}$ will be introduced in Section 3.

A *Brouwerian algebra* is an algebra $\mathbf{B} = \langle B; \wedge, \vee, \to \rangle$ such that $\langle B; \wedge, \vee \rangle$ is a lattice with order $\leq$ and $\to$ is the residuum of $\wedge$, that is, $a \wedge b \leq c$ iff $a \leq b \to c$, for all $a, b, c \in B$. Brouwerian algebras are precisely the bottom-free subreducts of Heyting algebras, the algebraic counterpart of intuitionistic logic. Given a Brouwerian algebra $\mathbf{B} = \langle B; \wedge, \vee, \to \rangle$, we say that a unary operator $\Box : B \to B$ is a *nucleus* if, for all $a, b \in B$, we have (i) $\Box(a \wedge b) = \Box a \wedge \Box b$ and (ii) $a \leq \Box a = \Box\Box a$. We shall refer to an algebra $\mathbf{B} = \langle B; \wedge, \vee, \to, \Box \rangle$ as a *nuclear Brouwerian algebra* [Rivieccio 2022, Def. 2.1].

Definition 1 ([Rivieccio 2022], Def. 2.2)**.** Let $\mathbf{B} = \langle B; \wedge, \vee, \to, \Box \rangle$ be a nuclear Brouwerian algebra. The algebra $\mathbf{B}^{\bowtie} = \langle B \times B; \wedge, \vee, \to, \sim \rangle$ is defined as follows. For all $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \in B \times B$,

$$\begin{aligned}
\sim\langle a_1, a_2 \rangle &= \langle a_2, \Box a_1 \rangle \\
\langle a_1, a_2 \rangle \wedge \langle b_1, b_2 \rangle &= \langle a_1 \wedge b_1, \Box(a_2 \vee b_2) \rangle \\
\langle a_1, a_2 \rangle \vee \langle b_1, b_2 \rangle &= \langle a_1 \vee b_1, a_2 \wedge b_2 \rangle \\
\langle a_1, a_2 \rangle \to \langle b_1, b_2 \rangle &= \langle a_1 \to b_1, \Box a_1 \wedge b_2 \rangle.
\end{aligned}$$

A *quasi-N4 twist-structure* $\mathbf{A}$ *over* $\mathbf{B}$ is a subalgebra of $\mathbf{B}^{\bowtie}$ satisfying the following properties: $\pi_1[A] = B$ and $\Box a_2 = a_2$ for all $\langle a_1, a_2 \rangle \in A$, where π_1 denote the first projection function.

The preceding definition provides a "concrete" way of producing examples of quasi-N4-lattices (see e.g. [Rivieccio 2022, Example 2.6]). In fact, as we shall see (Theorem 1), all quasi-N4-lattices may be obtained in this way.

Given an algebra $\mathbf{A}$ having an operation $\to$ and elements $a, b \in A$, we abbreviate $|a| := a \to a$, and define the relations $\equiv$ and $\preceq$ as follows. We let $a \preceq b$ iff $a \to b = |a \to b|$, and $\equiv := \preceq \cap (\preceq)^{-1}$. Thus one has $a \equiv b$ iff ($a \preceq b$ and $b \preceq a$).

Definition 2 ([Rivieccio 2022], Def. 3.2)**.** A *quasi-N4-lattice (QN4-lattice)* is an algebra $\mathbf{A} = \langle A; \wedge, \vee, \to, \sim \rangle$ of type $\langle 2, 2, 2, 1 \rangle$ satisfying the following properties:

(QN4a) The reduct $\langle A; \wedge, \vee \rangle$ is a distributive lattice with lattice order $\leq$.

(QN4b) The relation $\equiv$ is a congruence on the reduct $\langle A; \wedge, \vee, \to \rangle$ and the quotient $B(\mathbf{A}) = \langle A; \wedge, \vee, \to \rangle / {\equiv}$ is a Brouwerian algebra. Moreover, the operator $\Box$ given by $\Box[a] := {\sim}{\sim} a/{\equiv}$ for all $a \in A$ is a nucleus, so the algebra $\langle B(\mathbf{A}), \Box \rangle$ is a nuclear Brouwerian algebra.

(QN4c) For all $a, b \in A$, it holds that $a \leq b$ iff $a \preceq b$ and ${\sim} b \preceq {\sim} a$.

(QN4d) For all $a, b \in A$, it holds that ${\sim}(a \to b) \equiv {\sim}{\sim}(a \wedge {\sim} b)$.

(QN4e) For all $a, b \in A$,

(QN4e.1) $a \leq {\sim}{\sim} a$.

(QN4e.2) ${\sim} a = {\sim}{\sim}{\sim} a$.

(QN4e.3) ${\sim}(a \vee b) = {\sim} a \wedge {\sim} b$.

(QN4e.4) ${\sim}{\sim} a \wedge {\sim}{\sim} b = {\sim}{\sim}(a \wedge b)$.

The preceding definition is a straightforward generalization of Odintsov's definition of N4-lattices [Odintsov 2003]; indeed, a quasi-N4-lattice $\mathbf{A}$ is an N4-lattice if and only if $\mathbf{A}$ is involutive, that is, ${\sim}{\sim} a \leq a$ for all $a \in A$ [Rivieccio 2022, Prop. 3.8].

Theorem 1 ([Rivieccio 2022], Thm. 3.3)**.** Every quasi-N4-lattice $\mathbf{A}$ is isomorphic to a twist-structure over $\langle B(\mathbf{A}), \Box \rangle$ by the map $\iota : A \to A/{\equiv} \times A/{\equiv}$ given, for all $a \in A$, by $\iota(a) := \langle a/{\equiv}, {\sim} a/{\equiv} \rangle$.

The non-equational presentation of QN4-lattices given in Definition 2 can be replaced with an equational one, entailing that QN4-lattices form a variety of algebras.

Proposition 1 ([Rivieccio 2022], Prop. 3.7)**.** Items **(QN4b)** and **(QN4c)** in Definition 2 can be equivalently replaced by the following identities:

1. $|x| \to y \approx y$.
2. $(x \wedge y) \to x \approx |(x \wedge y) \to x|$.
3. $(x \wedge y) \to z \approx x \to (y \to z)$.
4. $(x \Leftrightarrow y) \to x \approx (x \Leftrightarrow y) \to y$.
5. $(x \vee y) \to z \approx (x \to z) \wedge (y \to z)$.
6. $x \to (y \wedge z) \approx (x \to y) \wedge (x \to z)$.
7. $(x \to y) \wedge (y \to z) \preceq x \to z$.
8. $x \to y \preceq x \to (y \vee z)$.
9. $x \to (y \to z) \approx (x \to y) \to (x \to z)$.
10. $x \to y \preceq {\sim}{\sim} x \to {\sim}{\sim} y$.

3. The logic of Quasi-N4-lattices

In this section we look at the logical counterpart of Quasi-N4-lattices, which is the logic $\mathbf{L}_{\mathrm{QN4}}$ introduced via a Hilbert-style calculus in [Lima Neto et al. 2022].

Fix a denumerable set $\mathcal{P}$ of propositional variables, and let $p \in \mathcal{P}$. The language of $\mathbf{L}_{\mathrm{QN4}}$ is defined recursively as follows:

$$\alpha ::= p \mid {\sim} \alpha \mid (\alpha \wedge \alpha) \mid (\alpha \vee \alpha) \mid (\alpha \to \alpha)$$

To simplify the notation, we shall henceforth omit the outmost parenthesis. We use $F_{\mathcal{P}}$ to denote the set of all formulas. A *logic* is defined as a finitary and substitution-invariant consequence relation $\vdash \subseteq \wp(F_{\mathcal{P}}) \times F_{\mathcal{P}}$ determined by a Hilbert-style calculus in the usual way. The calculus for $\mathbf{L}_{\mathrm{QN4}}$ consists of the following axiom schemes together with the single inference rule of *modus ponens* (MP): $p, p \to q \vdash q$.

$\mathbf{Ax1}$ $p \to (q \to p)$
$\mathbf{Ax2}$ $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$
$\mathbf{Ax3}$ $(p \land q) \to p$
$\mathbf{Ax4}$ $(p \land q) \to q$
$\mathbf{Ax5}$ $(p \to q) \to ((p \to r) \to (p \to (q \land r)))$
$\mathbf{Ax6}$ $p \to (p \lor q)$
$\mathbf{Ax7}$ $q \to (p \lor q)$
$\mathbf{Ax8}$ $(p \to r) \to ((q \to r) \to ((p \lor q) \to r))$
$\mathbf{Ax9}$ ${\sim}(p \lor q) \leftrightarrow ({\sim} p \land {\sim} q)$
$\mathbf{Ax10}$ ${\sim}(p \to q) \leftrightarrow {\sim}{\sim}(p \land {\sim} q)$
$\mathbf{Ax11}$ ${\sim}(p \land (q \land r)) \leftrightarrow {\sim}((p \land q) \land r)$
$\mathbf{Ax12}$ ${\sim}(p \land (q \lor r)) \leftrightarrow {\sim}((p \land q) \lor (p \land r))$
$\mathbf{Ax13}$ ${\sim}(p \lor (q \land r)) \leftrightarrow {\sim}((p \lor q) \land (p \lor r))$
$\mathbf{Ax14}$ ${\sim}{\sim}(p \land q) \leftrightarrow ({\sim}{\sim} p \land {\sim}{\sim} q)$
$\mathbf{Ax15}$ $p \to {\sim}{\sim} p$
$\mathbf{Ax16}$ $p \to ({\sim} p \to {\sim}(p \to p))$
$\mathbf{Ax17}$ $(p \to q) \to ({\sim}{\sim} p \to {\sim}{\sim} q)$
$\mathbf{Ax18}$ ${\sim} p \to {\sim}(p \land q)$
$\mathbf{Ax19}$ ${\sim}(p \land q) \to {\sim}(q \land p)$
$\mathbf{Ax20}$ $({\sim} p \to {\sim} q) \to ({\sim}(p \land q) \to {\sim} q)$
$\mathbf{Ax21}$ $({\sim} p \to {\sim} q) \to (({\sim} r \to {\sim} s) \to ({\sim}(p \land r) \to {\sim}(q \land s)))$
$\mathbf{Ax22}$ ${\sim}{\sim}{\sim} p \to {\sim} p.$

Recall that $\mathbf{Ax1}$-$\mathbf{Ax8}$ (together with *modus ponens*) constitute an axiomatization of the negation-free fragment of intuitionistic logic. In consequence, $\mathbf{L}_{\mathrm{QN4}}$ enjoys the classical Deduction Theorem: $\Gamma, \alpha \vdash \beta$ is equivalent to $\Gamma \vdash \alpha \to \beta$.

For an algebraizable logic $\mathbf{L}$ [Font 2016, Def. 3.11], we say that $\mathbf{L}$ is *finitely algebraizable* when the set of equivalence formulas is finite, and we say that $\mathbf{L}$ is *BP-algebraizable* when $\mathbf{L}$ is finitely algebraizable and the set of defining identities is finite. Let us abbreviate $x \Rightarrow y := (x \to y) \land ({\sim} y \to {\sim} x)$ and $x \Leftrightarrow y := (x \Rightarrow y) \land (y \Rightarrow x)$. The following result is an easy rephrasing of [Lima Neto et al. 2022, Thm. 4].

Theorem 2. $\mathbf{L}_{\mathrm{QN4}}$ is BP-algebraizable with defining identity $E(\alpha) := \alpha \approx |\alpha|$ and equivalence formula $\Delta(\alpha, \beta) := \alpha \Leftrightarrow \beta$.

By Theorem 2, we can obtain an axiomatization of the equivalent quasi-variety semantics $\mathrm{Alg}^*(\mathbf{L}_{\mathrm{QN4}})$ of $\mathbf{L}_{\mathrm{QN4}}$ as follows.

Definition 3. An $\mathrm{Alg}^*(\mathbf{L}_{\mathrm{QN4}})$-algebra is a structure $\mathbf{A} = \langle A; \land, \lor, \to, {\sim} \rangle$ which satisfies the following identities and quasi-identities:

1. $\alpha \approx |\alpha|$ for each axiom α of $\mathbf{L}_{\mathrm{QN4}}$.
2. $x \Leftrightarrow x \approx |x \Leftrightarrow x|$.
3. $x \Leftrightarrow y \approx |x \Leftrightarrow y|$ implies $x \approx y$.
4. $x \approx |x|$ and $x \to y \approx |x \to y|$ implies $y \approx |y|$.

As showed in [Lima Neto et al. 2022, Cor. 1], the class of algebras introduced in Definition 3 coincides with the variety of QN4-lattices (Definition 2), that is, $\mathrm{Alg}^*(\mathbf{L}_{\mathrm{QN4}}) = \mathsf{QN4}$.

4. Filters and congruence properties

In this section we look at filters of QN4-lattices and state a number of congruence-theoretic properties of this class of algebras; some of them are immediate consequences of the above-mentioned algebraizability result, but all are also proven directly in [Rivieccio 2022].

Recall that a *filter* of a Brouwerian algebra $\mathbf{B}$ is a (non-empty) lattice filter of the underlying lattice or, equivalently, a set $F \subseteq B$ that is non-empty and closed under *modus ponens*, meaning that $a, a \to b \in F$ entail $b \in F$ for all $a, b \in B$. The (po)set of all filters of a Brouwerian algebra $\mathbf{B}$ will be denoted by $Fi(\mathbf{B})$. Similarly, given a quasi-N4-lattice $\mathbf{A}$ and $F \subseteq A$, we shall say that F is an *(implicative) filter* if (i) $|a| \in F$ for all $a \in A$, and (ii) F is closed under *modus ponens* (if $a, a \to b \in A$, then $b \in F$, for all $a, b \in A$). The (po)set of all filters of a quasi-N4-lattice $\mathbf{A}$ is denoted by $Fi(\mathbf{A})$.

Theorem 3 ([Rivieccio 2022], Thm. 4.2)**.** For every QN4-lattice $\mathbf{A} \leq \mathbf{B}^{\bowtie}$, the first-coordinate projection map π_1 is a complete order isomorphism between $Fi(\mathbf{A})$ and $Fi(\mathbf{B})$.

Given a QN4-lattice $\mathbf{A}$ and a congruence $\theta \in Con(\mathbf{A})$, we define $F_\theta := \{a \in A : \langle a, |a| \rangle \in \theta\}$. Conversely, for each $F \in Fi(\mathbf{A})$, we let:

$$\theta_F := \{\langle a, b \rangle \in A \times A : a \Leftrightarrow b \in F\}.$$

Theorem 4 ([Rivieccio 2022], Thm. 4.4)**.** The maps given by $\theta \mapsto F_\theta$ and $F \mapsto \theta_F$ establish a complete order isomorphism between $Fi(\mathbf{A})$ and $Con(\mathbf{A})$.

It is well known that the nucleus does not alter the congruences of a Brouwerian algebra [Rivieccio 2022, Lemma 4.6]; thus we have the following result.

Corollary 1 ([Rivieccio 2022], Cor. 4.6)**.** Let $\mathbf{A} \leq \mathbf{B}^{\bowtie}$ be a quasi-N4-lattice, where $\mathbf{B} = \langle B, \wedge, \vee, \to, \Box \rangle$. Then $Con(\mathbf{A}) \cong Con(\mathbf{B}) = Con(\langle B, \wedge, \vee, \to \rangle)$.

For the unexplained universal algebraic terms employed below, see [Burris and Sankappanavar 2012].

Corollary 2 ([Rivieccio 2022], Cor. 4.13)**.** Let $\mathbf{A} \leq \mathbf{B}^{\bowtie}$ be a quasi-N4-lattice, where $\mathbf{B} = \langle B, \wedge, \vee, \to, \Box \rangle$. The following are equivalent:

1. $\mathbf{A}$ is directly indecomposable (resp., subdirectly irreducible, simple).
2. $\langle B, \wedge, \vee, \to \rangle$ is a directly indecomposable (resp., subdirectly irreducible, simple) Brouwerian algebra.

The variety of QN4-lattices possesses a ternary deduction term and a quaternary deductive term in the sense of [Blok and Pigozzi 1994], as well as a Maltsev term witnessing congruence-permutability. This entails that quasi-N4-lattices enjoy equationally definable principal congruences and the strong version of the congruence extension property considered in [Blok and Pigozzi 1994, Def. 2.10].

Theorem 5 ([Rivieccio 2022], Thm. 4.9)**.** $q(x, y, z) := (x \Leftrightarrow y) \to z$ is a commutative ternary deduction term for QN4 in the sense of [Blok and Pigozzi 1994].

Applying Theorem 5, we note that, for every quasi-N4-lattice $\mathbf{A}$, the principal congruence generated by elements $a, b \in A$ is given by:

$$\theta(a,b) = \{\langle c,d\rangle : (a \Leftrightarrow b) \to c = (a \Leftrightarrow b) \to d\}.$$

Theorem 6 ([Rivieccio 2022], Thm. 4.10)**.** QN4 is congruence-permutable with Maltsev term:

$$p(x,y,z) := (((x \Rightarrow y) \wedge |z|) \Rightarrow z) \wedge (((z \Rightarrow y) \wedge |x|) \Rightarrow x).$$

As a variety of enriched lattices, quasi-N4-lattices are obviously congruence-distributive. Thus, the preceding theorem extends the result of [Spinks and Veroff 2018, Cor. 4.25] to our non-involutive setting.

Corollary 3 ([Rivieccio 2022], Cor. 4.11)**.** QN4 is arithmetical.

Corollary 4 ([Rivieccio 2022], Cor. 4.12)**.** QN4 has a quaternary deductive term:

$$t(x,y,z,w) := p(q(x,y,z), q(x,y,w), w),$$

where $p(x,y,z) = (((x \Rightarrow y) \wedge |z|) \Rightarrow z) \wedge (((z \Rightarrow y) \wedge |x|) \Rightarrow x)$ and $q(x,y,z) = (x \Leftrightarrow y) \to z$.

5. Ongoing and future research

As mentioned earlier, research on quasi-N4-lattices is necessarily at a preliminary stage, and only time will tell to what extent further investigations on this and related classes of algebras will prove fruitful. We mention below a few directions that appear to be of obvious interest.

Refining the twist construction. By Theorem 1, we know that we can identify an arbitrary quasi-N4-lattice $\mathbf{A}$ with a subalgebra of $\mathbf{B}^{\bowtie}$ for some nuclear Brouwerian algebra $\mathbf{B}$. This establishes a correspondence (which may be rephrased as an adjunction between suitably defined categories) between each nuclear Brouwerian algebra $\mathbf{B}$ and the family of quasi-N4-lattices that canonically embed into $\mathbf{B}$.

As shown in [Rivieccio 2022, Prop. 2.5], two further parameters ∇ and Δ (respectively, a lattice filter and an ideal of $\mathbf{B}$) are sufficient to uniquely determine a twist-algebra having the following set as underlying universe:

$$Tw(B, \nabla, \Delta) := \{\langle a_1, a_2\rangle \in B \times B : a_2 = \Box a_2,\ a_1 \vee a_2 \in \nabla,\ a_1 \wedge a_2 \in \Delta\}.$$

We thus have a one-to-one correspondence between triples (B, ∇, Δ) and quasi-N4-lattices, but we do not currently know whether *every* quasi-N4-lattice arises in this way. If the latter was true, then the correspondence would yield an equivalence between the algebraic category of quasi-N4-lattices and a category having as objects triples (B, ∇, Δ); this is indeed known to hold for N4-lattices [Rivieccio and Jansana 2021].

Ongoing research on the non-involutive twist construction suggests that the question may be settled at least for every quasi-N4-lattice $\mathbf{B}$ that possesses a residuated lattice structure, that is, further algebraic operations $\mathbf{1}$ and $*$ such that $\langle B, *, \mathbf{1}\rangle$ is a (commutative) monoid and the pair $(*, \Rightarrow)$ is residuated. The resulting class of algebras, which we may dub *residuated quasi-N4-lattices*, is also relevant to the research direction discussed below.

Quasi-N4-lattices as residuated structures. As observed in [Rivieccio 2022, Rem. 3.5], the so-called weak implication $\rightarrow$ is definable in quasi-N4-lattices using the so-called strong one $\Rightarrow$ and the conjunction $\wedge$. Recalling that the strong implication is the 'substructural implication' of (quasi-)N4-lattices, this suggests that it is possible to axiomatize the class of quasi-N4-lattices and the corresponding logic in the language $\{\wedge, \vee, \Rightarrow, \sim\}$, which in turn may allow us to establish a more direct comparison with (algebras of) relevance logics. However, given the defining term employed in [Rivieccio 2022, Rem. 3.5], we may expect the axiomatizations thus obtained to be rather unwieldy.

The picture may become more clear if we are willing to further expand the language of QN4 by introducing connectives corresponding to the above-mentioned monoid conjunction $*$ (and maybe the identity $\mathbf{1}$ as well); the resulting algebraic models will be a class of residuated structures, which one may hope study within the theory of *paraconsistent Nelson RW-algebras* developed in [Spinks and Veroff 2018].

Quasi-N4-lattices and relevant algebras. The paper [Galatos and Raftery 2015] introduced the variety of *generalized Sugihara monoids* as a non-involutive generalization of algebraic models of the relevant logic *R-mingle*, a class of algebras known as *Sugihara monoids*. One of the main results of Galatos and Raftery is that generalized Sugihara monoids are representable through a twist construction which has striking similarities with the one for quasi-N4-lattices. The factor algebras employed in their twist construction are in fact nuclear Brouwerian algebras that are also prelinear (i.e. representable as subdirect products of linearly ordered ones).

While the equational properties of the two above-mentioned classes of algebras suggest that a direct comparison between (generalized) Sugihara monoids and (quasi-) N4-lattices is not likely to prove fruitful, we speculate that the twist construction may be used to establish a meaningful connection. Indeed, since the twist representation is used in [Galatos and Raftery 2015] to establish a categorical equivalence between generalized Sugihara monoids and prelinear nuclear Brouwerian algebras, it may be possible to apply a similar strategy to quasi-N4-lattices, namely, single out a subcategory of (perhaps enriched) quasi-N4-lattices that may be proved to be equivalent as a category to the prelinear nuclear Brouwerian algebras considered in [Galatos and Raftery 2015]. An equivalence with generalized Sugihara monoids would then be obtained as an immediate corollary. We leave this as a final suggestion for further research.

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