Tableau Calculus for Only-knowing and Abduction: A Preliminary Report

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Abstract. *Introduced by Hector Levesque in the 1990s, the logic of only-knowing ($\mathcal{OL}$) has proven to be a rich framework for knowledge representation, a central topic in Artificial Intelligence. Since then, various extensions of $\mathcal{OL}$ have been explored by different authors. More recently, S. Molick and V. Belle extended $\mathcal{OL}$ to incorporate abductive reasoning, resulting in the logic of only-knowing and abduction ($\mathcal{AOL}$). In this paper, we present sound and complete tableau rules for $\mathcal{AOL}$, providing a preliminary step toward the development of an adequate proof-theoretic foundation for abductive reasoning within this extended logic.*

1. Introduction

The development of logic-based frameworks for knowledge representation is an important subject for modern logic and theoretical computer science. An influential framework for modeling epistemic reasoning in different scenarios was first introduced by Hector Levesque in [Levesque 1990]. Levesque's basic framework, named **Logic of Only-knowing** ($\mathcal{OL}$), was subsequently extended by different authors for handling multi-agent scenarios, the dynamics of update announcement and belief revision, among other relevant features for epistemic modelling (see [Belle and Lakemeyer 2010], [Levesque 1989], [Belle and Lakemeyer 2015]).

A novel extension of $\mathcal{OL}$ was recently introduced in [Molick and Belle 2025] to express abductive reasoning within the bounds of the background knowledge of an agent, a framework called **Logic of Only-knowing and Abduction** ($\mathcal{AOL}$). Abductive reasoning is a central component for contemporary accounts of knowledge representation in artificial intelligence. It is often described as a kind of "backward reasoning". While deductive reasoning seeks to determine whether a certain conclusion φ follows from a background knowledge Δ (written $\Delta \vdash \varphi$), abductive reasoning consists of working from the pair $\langle \Delta, \varphi \rangle$ (called an *abduction problem*) such that $\Delta \not\vdash \varphi$ and seeking for a formula α (called an *explanation* for the event φ) such that $\Delta, \alpha \vdash \varphi$.

In $\mathcal{AOL}$, abductive reasoning is expressed by the modality $\mathbf{A}\varphi$ capable of expressing that the "formula φ is inferred by abduction". The rationale is to treat abduction as a process of epistemic change, a topic initiated in [Aliseda 2000]. For this, the logic $\mathcal{AOL}$ validates core inferences of abductive reasoning for propositional modal languages such as $\models (O(\phi \to \psi) \wedge O\psi)) \to O\phi$ or $O(\phi \to \psi), O\psi \models O\phi$ (where $\mathbf{O}\varphi$ expresses that the agent "only knows φ") via semantic means. While the authors in [Molick and Belle 2025] explored only semantical properties of the system, the purpose of this paper is to introduce a sound and complete tableau rules for the logic of only-knowing and abduction. The result represents a first step toward the development of tableaux systems suitable for abductive reasoning in the sense proposed by [Aliseda 2000] or [Nepomuceno-Fernández 2002].

2. Preliminaries

Let $\mathcal{L}$ be a propositional modal language with a countable set of propositional letters p, q, r..., a countable set of arbitrary formulas φ, ψ, δ ..., the classical operators $\neg$, $\wedge$, $\rightarrow$ and the modalities $\mathbf{K}$, $\mathbf{O}$ and $\mathbf{A}$. The set of $\mathcal{L}$-formulas of the logic $\mathcal{AOL}$ is recursively defined in the following way:

$$\varphi :== p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathbf{K}\varphi \mid \mathbf{O}\varphi \mid \mathbf{A}\varphi$$

We will follow Levesque's presentation ([Levesque 1990]) and let $\mathbf{O}\varphi$ to be read as "the agent only knows φ" and $\mathbf{K}\varphi$ to be read as "the agent knows or believes φ". In addition, $\mathbf{A}\varphi$ should be read as "the agent knows φ by abduction". Accordingly, all propositional formulas will be called **objective**, boolean formulas preceded by any of the three modal operators will be called **subjective**. A formula will be called **abductive** if it is a boolean formula preceded only by the modal operator $\mathbf{A}$.

According to Levesque's semantics, an epistemic situation is a pair $(\mathcal{W}, w)$, where $\mathcal{W}$ is a set of epistemic states and w has the usual truth-assignment to primitive formulas. The resulting Kripke's semantics is defined as follows:

Definition 2.1. *([Molick and Belle 2025]) Where $\mathcal{L}$ is a modal language, a **Kripke model** for $\mathcal{L}$ is a tuple $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, v \rangle$, where $\mathcal{W}$ is a set of epistemic states, $\mathcal{R}$ is a binary relation over $\mathcal{W}$ and v is a mapping that assigns to each atom of $\mathcal{L}$ a subset of $\mathcal{W}$.*

(i) $(\mathcal{M}, w) \models p$ *iff* $w \in v(p)$ *and* p *is an atom.*
(ii) $(\mathcal{M}, w) \models \neg\varphi$ *iff* $(\mathcal{M}, w) \not\models \varphi$.
(iii) $(\mathcal{M}, w) \models \varphi \wedge \psi$ *iff* $(\mathcal{M}, w) \models \varphi$ *or* $(\mathcal{M}, w) \models \psi$.
(iv) $(\mathcal{M}, w) \models \varphi \rightarrow \psi$ *iff* $(\mathcal{M}, w) \not\models \varphi$ *or* $(\mathcal{M}, w) \models \psi$
(v) $(\mathcal{M}, w) \models \mathbf{K}\varphi$ *iff* $(\mathcal{M}, w') \models \varphi$ *for all* $w' \in \mathcal{M}$ *such that* wRw'.
(vi) $(\mathcal{M}, w) \models \mathbf{O}\varphi$ *if for all states* $w' \in \mathcal{W}$: wRw' *iff* $(\mathcal{M}, w') \models \varphi$.
(vii) $(\mathcal{M}, w) \models \mathbf{A}\varphi$ *iff* $\exists\alpha : (\mathcal{M}, w) \models \mathbf{O}\alpha$ *and* $(\mathcal{M}, w) \models \mathbf{K}(\varphi \rightarrow \alpha)$.

Definition 2.2. *(Local validity) We shall write $(\mathcal{M}, w) \models \varphi$ to denote that the modal formula φ is valid at the state w of a model $\mathcal{M}$. We shal write '$\Gamma \models \varphi$' to denote that every model $(\mathcal{M}, w)$ such that $(\mathcal{M}, w) \models \gamma$ (for every $\gamma \in \Gamma$) implies $(\mathcal{M}, w) \models \varphi$.*

Definition 2.3. *(Global validity) A formula φ is valid with respect to a class $\mathcal{F}$ of models $\mathcal{M}$ (written $\models^{\mathcal{F}} \varphi$) if $(\mathcal{M}, w) \models \varphi$ for every model $\mathcal{M} \in \mathcal{F}$.*

Definition 2.4. *(Abductive explanation) A formula α will be called an **explanation** for the abduction problem $\langle \Theta, \varphi \rangle$ if α is an abductive formula and $\Theta \cup \{\alpha\} \models \varphi$.*

As explained by the authors in [Molick and Belle 2025], an useful application of the abductive modality $\mathbf{A}$ is to employ abductive reasoning in accordance with the background knowledge of the agent[1]. Thus, clause (vii) of the semantics guarantees that the formula explanation discovered by the agent is within the bounds of her background knowledge, as the following example illustrates.

Example 1. *Let Θ denote a set of diagnostic principles, F denote the set of symptoms of a patient, and E be the set of possible diagnostic explanations. Consider the following medical situation*

$$\Theta = \{\mathbf{O}(cold \to cough),$$
$$\mathbf{O}(flu \to (cough \wedge fever)),$$
$$\mathbf{O}(pneumonia \to (chest\_pain \wedge cough \wedge fever))\}$$
$$F = \{fever, cough\}$$
$$E = \{flu\}.$$

In this example the best abductive explanation for the patient's symptom is that it is caused by the condition flu. According to the semantics of the abduction operator $\mathbf{A}$, the diagnostician obtains the explanation $\varphi = flu$. Since both $\mathbf{O}\alpha$ and $\mathbf{K}(\varphi \to \alpha)$ hold, the diagnostician can conclude $\mathbf{A}\varphi$, which in this case corresponds to flu. As a consequence, the doctor does not directly know that the patient has the flu, but abductively infers it based on the symptom $cough \wedge fever$. The combination of $\mathbf{K}$ and $\mathbf{O}$ expresses the relation between the only-known facts ($\mathbf{O}$) acknowledged by the agent and the possible explanations expressed by $\mathbf{K}$. The abductive explanation $\mathbf{A}$ is derived in accordance with the hypotheses for which the agent is aware of explaining the event α.

2.1. Tableaux system

In this section, we introduce a set of tableau rules for the logic $\mathcal{AOL}$ (in the style presented in [Priest 2008]) . Unlike standard modal logics, $\mathcal{AOL}$ incorporates epistemic operators that capture both the agent's explicit knowledge and the restrictions imposed by only-knowing specific information. The rules introduced in this section are designed to reflect the semantics of the three central modal operators in $\mathcal{AOL}$: the knowledge operator ($\mathbf{K}$), the only-knowing operator ($\mathbf{O}$), and the abduction operator ($\mathbf{A}$). We start by introducing the basic definitions:

Definition 2.5. *A **modal tableau tree** is a structured sequence of **nodes**, where each node is any finite set of formulas endowed with a state w. The node at the top of a tree is called the **root**. The nodes at the bottom are called **tips**. A **branch** is a maximal sequence of nodes, where each node is obtained by applying a tableau rule to a node.*

Definition 2.6. *The **initial list** of a tableau tree is the single branch in which occur the premises and the negation of the conclusion, where each node comes endowed with both a formula and an initial state.*

Definition 2.7. *A tableau is **complete** iff every rule that can be applied has been applied.*

[1] The reader may check [Molick and Belle 2025] to see other semantic properties that guarantee this behavior.

Definition 2.8. *A branch is* ***closed*** *iff there are formulas of the form φ and $\neg\varphi$ on two of its nodes labeled with the same state; otherwise it is* ***open****. (A closed branch will be indicated by $\times$). A tableau is* ***closed*** *iff every branch is closed; otherwise it is open.*

Definition 2.9. *We will say that the formula φ is a* ***proof-theoretic*** *consequence of the set of formulas Γ written $(\Gamma \vdash \varphi)$ iff there is a complete tableau whose initial list is closed and starts with the formulas of Γ and the negation of φ. We will also write $\Gamma \models \varphi$ to denote that the formula φ is a* ***semantic*** *consequence of the set of formulas Γ.*

The tableau rules in Priest's presentation come endowed with a formula and an index number to indicate the world at which the formula is valid. Thus, ψ, w_1 indicates that the formula is valid at state w_1 and $w_1 r w_2$ indicates that w_2 is accessible from w_1.Furthermore, each connective has a pair of rules, one of which applies to its negated form. We introduce below all rules for our tableaux system[2]:

2.2. Rules for Conjunction

Conjunction Rule

$$\cfrac{\cfrac{\phi \wedge \psi, w_i}{\phi, w_i}}{\psi, w_i}$$

Negated Conjunction Rule

$$\frac{\neg(\phi \wedge \psi), w_i}{\neg\phi, w_i \mid \neg\psi, w_i}$$

2.3. Rules for Disjunction

Disjunction Rule

$$\frac{\phi \vee \psi, w_i}{\phi, w_i \mid \psi, w_i}$$

Negated Disjunction Rule

$$\cfrac{\cfrac{\neg(\phi \vee \psi), w_i}{\neg\phi, w_i}}{\neg\psi, w_i}$$

2.4. Rules for Conditional

Conditional Rule

$$\frac{\phi \rightarrow \psi, w_i}{\neg\phi, w_i \mid \psi, w_i}$$

Negated Conditional Rule

$$\cfrac{\cfrac{\neg(\phi \rightarrow \psi), w_i}{\phi, w_i}}{\neg\psi, w_i}$$

The rules for the modal operators are the following:

2.5. Rules for Knowledge

Knowledge Rule

$$\frac{\mathbf{K}\phi, w_i \text{ and } w_i R w_j}{\phi, w_j}$$

Negated Knowledge Rule

$$\cfrac{\cfrac{\neg\mathbf{K}\phi, w_i}{w_i R w_j}}{\neg\phi, w_j}$$

[2]The '|' denotes an or, i.e. a bifurcation in the branch.

The knowledge rule states that if $\mathbf{K}\phi$ is in a state w_i, then ϕ can be introduced in all states w_j such that $w_i R w_j$. In a different situation, the negated knowledge rule allows us to introduce $\neg\phi$ only in new states w_j.

2.6. Rules for Only-knowing

Only-Knowing Rule

$$\frac{\mathbf{O}\varphi, w_i}{\phi, w_j \text{ and } w_i R w_j}$$

Negated Only-knowing Rule

$$\frac{\neg\mathbf{O}\varphi, w_i}{\neg\phi, w_j \text{ and } w_i R w_j \mid \varphi, w_j \text{ and } \neg w_i R w_j}$$

The only-knowing rules states that if $\mathbf{O}\phi$ holds at a state w_i, then ϕ must hold in all accessible states. The negated rule works in terms of the biconditional, i.e., either a state validates φ or is not accessible. In spite of its similarity with the knowledge rule, it is important to note that only the knowledge rule allows one to create new states.

2.7. Rules for Abduction

Abduction Rule

$$\frac{\mathbf{A}\varphi, w_i}{\mathbf{O}\alpha, w_i,\ \ \mathbf{K}(\alpha \to \varphi), w_i}$$

Negated Abduction Rule

$$\frac{\neg\mathbf{A}\varphi, w_i}{\neg\mathbf{O}\alpha, w_i,\ \mid \neg\mathbf{K}(\alpha \to \varphi), w_i}$$

The abduction rule states that if $\mathbf{A}\phi$ holds at a state w, one can introduce an abductive assumption α such that both $\mathbf{O}\alpha$ and $\mathbf{K}(\alpha \to \phi)$ hold in w[3]. In contrast, the negated abduction rule states that either $\mathbf{O}$ or $\mathbf{K}$ fails at a state w for any abductive assumption α. In the next section, we introduce the proofs of soundness and completeness.

3. Soundness and Completeness

The soundness and completeness proofs follow the standard practice and presentation for tableaux systems as developed in [Priest 2008]. In the following, we present the relevant parts of the proof.

Definition 3.1. *Let $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, v\rangle$ be a kripke model, and b be any branch of a tableau. We say that $\mathcal{M}$ is faithful to b iff there is a map $f : \mathbb{N} \to \mathcal{W}$ such that*
(i) For every node φ, w_i on b, then φ is true at $f(w_i)$ in $\mathcal{M}$, and
(ii) If $w_1 R w_2$ on b, then $f(w_1)Rf(w_2)$ in $\mathcal{M}$.
We say that the map f shows $\mathcal{M}$ to be ***faithful*** *to b.*

Lemma 1. *(Soundness lemma) Given a branch b and a kripke model $\mathcal{M}$, if $\mathcal{M}$ is faithful to b, and a tableau rule is applied to it, then it produces an extension b' such that $\mathcal{M}$ is also faithful to b'.*

[3]The problem of finding an adequate α for any abduction problem $\langle\Theta, \phi\rangle$ is still open. Thus, a central assumption of our tableaux system is that it handles only abduction problems for which there is an explanation α.

Proof. The proof proceeds by induction on the size of the branch, analyzing the application of each rule that may extend the branch under consideration. We show only the case for the modal operators[4]. Let f be a function that shows a model $\mathcal{M}$ to be faithful to a branch b.

(**K**-case) Suppose that $\mathbf{K}\varphi, w_i$ appears in b and we apply the knowledge rule. According to this rule, we know that $w_i R w_j$ and obtain an extended branch with a state w_j at which ϕ holds. Finally, since $\mathcal{M}$ is faithful to b, we know that $\mathbf{K}\varphi, f(w_i)$ appears in b. Again, by the knowledge rule, we get an extended branch b' with a state $f(w_j)$ at which ϕ holds. As a consequence, $\mathcal{M}$ is faithful to b'. For the negated knowledge rule, suppose that $\neg\mathbf{K}\varphi, w_i$ appears in a branch b. According to the negated knowledge rule, one is allowed to create a novel state w_j such that $w_i R w_j$ and $\neg\phi$ holds at w_j. Now, given that $\mathcal{M}$ is faithful to b, we know that $\neg\mathbf{K}\varphi, f(w_i)$ appears in b. Moreover, by the negated rule, we obtain an extended branch with a state $f(w_j)$ such that $f(w_i) R f(w_j)$ and $\neg\phi$ holds. Finally, $\mathcal{M}$ is faithful to b'.

(**A**-case) Suppose that $\mathbf{A}\varphi, w$ appears in b and we apply the abduction rule and get an extended branch containing $\mathbf{O}\alpha, w$ and $\mathbf{K}(\alpha \to \varphi), w$. Since $\mathcal{M}$ is faithful to b, $\mathbf{A}\varphi$ is true at $f(w)$. Hence $\mathbf{O}\alpha$ and $\mathbf{K}(\alpha \to \varphi)$ are true at $f(w)$. For the negated abduction rule, $\neg\mathbf{A}\varphi, w$ appears in b and we apply its corresponding rule. Hence, we get two branches, one extending b with $\neg\mathbf{O}\alpha, w$ (left) and the other extending b with $\neg\mathbf{K}(\alpha \to \varphi), w$ (right). For each branch apply, respectively, the negated only-knowing and negated knowledge rule[5]. For both cases, b is extended to a branch b' that is faithful to $\mathcal{M}$. In the first case f is faithful to the left branch; in the second case, f is faithful to the right branch. The rest of the proof for the negated abduction rule and the **O** operator follows by analogous reasoning. □

Theorem 1 (Soundness). *For finite Σ, if $\Sigma \vdash \varphi$ then $\Sigma \models \varphi$.*

Proof. Assume that $\Sigma \not\models \varphi$. By definition of validity, there is a Kripke model such that $(\mathcal{M}, w) \models \Sigma$ and $(\mathcal{M}, w) \not\models \phi$. Now assume there is a completed tableau such that $\mathcal{M}$ is faithful to the initial list. By the Soundness lemma, any extension of it remains faithful. As a result, the tableau remains open and our desired conclusion $\Sigma \nvdash \varphi$ follows. □

Definition 3.2. *Let b be an open branch of a tableau. We will call a model $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, v \rangle$ the* ***model induced*** *by b iff $\mathcal{M}$ is defined as follows:*
(i) $\mathcal{W} = \{w_i : i \text{ occurs on } b\}$
(ii) $w_i R w_j$ iff iRj occurs on b
(iii) If p, i occurs on b, then $v_{w_i}(p) = 1$;
if $\neg p, i$ occurs on b, then $v_{w_i}(p) = 0$.

Lemma 2. *(Completeness lemma) Where b is an open complete branch of a tableau, let $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, v \rangle$ be the model induced by b. Hence:*
(i) if ϕ, i is on b, then ϕ is true at w_i
(ii) if $\neg\phi, i$ is on b, then $\neg\phi$ is false at w_i.

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[4]The reader may check [Priest 2008] for a detailed description of the propositional operators.

[5]The negated knowledge rule requires the creation of a novel state, say w_j such that $w_i R w$. For this, it is sufficient to note that one can extend f to a map f' such that f' is equivalent to f and $f'(j) = w$.

Proof. The proof is by induction on the structure of φ. We show only the modal cases. Consider the case ϕ is of the form $\mathbf{K}\psi$. Suppose that $\mathbf{K}\psi$ appears in b. Since b is complete, the knowledge rule was applied. Therefore, ψ is in b. Finally, by induction hypothesis, $v(\psi) = 1$. Now suppose that $\neg\mathbf{K}\psi$ appears in b. By the negated knowledge rule, $\neg\psi$ is in b. Again, by induction hypothesis, $v(\psi) = 0$. The proof for the $\mathbf{O}$ operator follows by analogous reasoning. For the abduction operator $\mathbf{A}$, the semantic clause makes it reducible to the cases $\mathbf{K}$ and $\mathbf{O}$.

□

Theorem 2 (Completeness)**.** *For finite* Σ*, if* $\Sigma \models \varphi$ *then* $\Sigma \vdash \varphi$*.*

Proof. Suppose that $\Sigma \nvdash \phi$ and consider a completed open tableau based on it. The interpretation $\mathcal{M}$ induced by it gives us $(\mathcal{M}, w) \models \Sigma$ and $(\mathcal{M}, w) \not\models \phi$. Finally, by the completeness lemma and the definition of validity, $\Sigma \not\models \phi$ follows. □

4. Future work

This paper introduced a structured proof system based on tableaux rules for the Logic of Only-Knowing and Abduction ($\mathcal{AOL}$). Future work may explore extending our tableau semantics by developing dynamic tableau rules for creative abduction or resolution calculus for the logic of only-knowing and abduction[6].

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