A New Total Order for Triangular Fuzzy Numbers with an Application

Valentina Rosas[1], **Benjamín Bedregal**[2], **José Canumán**[3], **Roberto Díaz**[4], **Edmundo Mansilla**[5] **and Nicolás Zumelzu**[5]

[1]Carrera de Pedagogía en Matemática para Educación Media
Departamento de Educación y Humanidades – Universidad de Magallanes
Punta Arenas – Chile

[2]Departamento de Informática e Matemática Aplicada
Universidade Federal do Rio Grande do Norte – Natal, RN – Brazil

[3]Departamento de Ingeniería en Computación e Informática
Universidad de Magallanes – Punta Arenas – Chile

[4]Departamento de Ciencias Exactas
Universidad de Los Lagos – Osorno – Chile

[5]Departamento de Matemática y Física – Universidad de Magallanes
Punta Arenas – Chile

{valrosas,jose.canuman,edmundo.mansilla,nicolas.zumelzu,}@umag.cl

bedregal@dimap.ufrn.br,roberto.diaz@ulagos.cl

Abstract. *This work deals with the study of a new total order Triangular Fuzzy Numbers and arithmetic properties that are maintained in relation to the operations of addition and subtraction. Additionally, we present as example of application the shortest path solution for the Travelling Salesman Problem with fuzzy distances.*

Resumo. *Este trabalho trata do estudo de uma nova ordem total para números fuzzy triangulares e propriedades aritméticas que são mantidas em relação às operações de adição e subtração. Além disso, apresentamos como exemplo de aplicação a solução do caminho mais curto para o Problema do Caixeiro Viajante com distâncias fuzzy.*

1. Introduction

In [Zadeh 1965] was proposed the concept of fuzzy sets in order to formalize and model the ambiguities and imprecision inherent to some linguistic terms, like to downs, young and quick, associated to linguistic variables like temperature, age and speed. Fuzzy numbers, a special class of fuzzy sets, were proposed in [Zadeh 1965] to model a quantity which is imprecise, rather than exact as is the case of a real number. Besides, arithmetic operations for fuzzy numbers and their properties had been widely study (see [Dubois and Prade 1978, Wang and Wang 2014]). In particular, we only consider fuzzy numbers with a triangular shape, which are called of Triangular Fuzzy Numbers, TFN in short.

In this sense, besides to serious algebraic consequences, the impossibility of obtaining both multiplicative and additive inverses is an inconvenient when trying to solve equations, even if it is a simple one of the first degree, which makes it necessary to find methods to solve this problem.

There are several proposal for orders on fuzzy numbers in general [Buckley 2005, Wang and Wang 2014, Zumelzu et al. 2020] and for TFN in particular (see for example [Akyar et al. 2012, Asmus.T.C. et al. 2017, Nasseri and Behmanesh 2013]) and here we propose a new total order for TFNs which be compatible, in some sense, with the addition and subtraction. Thereby, we propose to retrieve some characteristics of fields using order properties. We also apply both, the addition and the total order on TFN, to solve the minimum path problem in fuzzy weighted graphs.

This paper is organized as follows: In Section 2, in addition to establishing the notation used, we recall some essential notions for the remaining sections. In Section 3 we see the more basic order on fuzzy numbers and properties of order are studied with arithmetic operations. The notion of orders for fuzzy numbers is applied in Section 4 including an algorithm used to find these shortest routes. Finally, Section 5 provides some final remarks.

2. Preliminaries

In this section, we provide the concepts of fuzzy number and the main tools that will be used to obtain the main results. Let $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, with $\mathbb{R}$ being the set of real numbers. The notation $<$, $>$, $\leq$ or $\geq$ stands for usual order of $\mathbb{R}$, and $[a,b]$, $]a,b]$, $[a,b[$ and $]a,b[$ the intervals of $\mathbb{R}$ closed, right-closed, right-open, and open respectively (see [Rudin 1964]).

Definition 2.1. *[Klir and Yuan 1995] A fuzzy set A on $\mathbb{R}$ is called a fuzzy number if it satisfies the following conditions*

(i) A is normal, i.e., $\sup A(x) = 1$*;*
(ii) ${}^{A}\!/\!_{\alpha}$ is a closed interval for every $\alpha \in]0,1]$;
(iii) the support of A is bounded,

where ${}^{A}\!/\!_{\alpha}$ is the α-level (or α-cut) of A.

A fuzzy number A is crisp if there exists $r \in \mathbb{R}$ such that $A(r) = 1$ and $A(x) = 0$ for each $x \neq r$. In this case we will denote A by $\widetilde{r}$. Finally, $\mathcal{F}(\mathbb{R})$ will denote the set of all fuzzy numbers.

Proposition 2.1. *[Dubois and Prade 1978] Let $A, B \in \mathcal{F}(\mathbb{R})$ then the fuzzy sets $A + B$ and $A - B$ defined by*

$$(A+B)(x) = \sup_{x=y+z} \min\{A(y), B(z)\} \text{ and } (A-B)(x) = \sup_{x=y-z} \min\{A(y), B(z)\},$$

for each $x \in \mathbb{R}$ is a fuzzy number.

Definition 2.2. *[Klir and Yuan 1995] A fuzzy number A, is called a triangular fuzzy number,* TFN *in short, if there is $(a,b,c) \in \mathbb{R}^3$ such that $a \leq b \leq c$ and*

$$A(x) = \begin{cases} 1, & \text{if } x = b, \\ \frac{x-a}{b-a}, & \text{if } x \in]a,b[, \\ \frac{c-x}{c-b}, & \text{if } x \in]b,c[, \\ 0, & \text{other cases.} \end{cases}$$

Proposition 2.2. *[Dubois and Prade 1978] Let* $A = (a_1, b_1, c_1)$ *and* $B = (a_2, b_2, c_2)$ *be TFNs. Then*

$$A + B = (a_1 + a_2, b_1 + b_2, c_1 + c_2) \text{ and } A - B = (a_1 - c_2, b_1 - b_2, c_1 - a_2).$$

In the following, we consider the notion of fuzzy weighted graphs, which are weighted graphs (see [Bondy and Murty 1976]) where the weights are fuzzy numbers.

Definition 2.3. *[Cornelis et al. 2004] A fuzzy weighted graph is a triple* $G = \langle V, E, c \rangle$ *where* V *is a set whose elements are called vertices,* $E \subseteq V \times V$ *is a set of edges and* $c : E \to \mathcal{F}(\mathbb{R})$ *is the cost (or weight) function. Given* $v, u \in V$*, a* (v, u)*-path in* G *is a finite and non empty sequence of edges* $p = (e_1, \ldots, e_n) = ((v_1, u_1), \ldots, (v_n, u_n))$ *such that* $v_i = u_{i-1}$ *for each* $i = 2, \ldots, n$, $v_1 = v$ *and* $u_n = u$. *A* (v, u)*-path is a cycle if* $v = u$.

The cost of a (v, u)-path $p = (e_1, \ldots, e_n)$, denoted by $c(p)$ is given by the addition of the cost of each edge in the path, i.e. $c(p) = \sum_{i=1}^{n} c(e_i)$. Given a pair of vertices (v, u) in a fuzzy weighted graph G there may exist several or no (v, u)-path.

3. Total order on triangular fuzzy numbers

In this section, we prove that the order to propose is a total order, we also study properties that it verifies with addition and subtraction operations.

Definition 3.1. *Let* (a_1, b_1, c_1) *and* (a_2, b_2, c_2) *two triangular fuzzy numbers.*

$$(a_1, b_1, c_1) \leq_{OT} (a_2, b_2, c_2) \text{ if and only if } \begin{cases} b_1 < b_2, \text{ or} \\ b_1 = b_2 \text{ and } a_1 < a_2, \text{ or} \\ b_1 = b_2 \text{ and } a_1 = a_2 \text{ and } c_1 \leq c_2. \end{cases}$$

Proposition 3.1. *The relation* $\leq_{OT}$ *is a total order.*

Proof. Let $(a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3) \in \mathcal{F}(\mathbb{R})$.

1. Reflexivity: Straightforward from Definition 3.1.

2. Antisymmetry: Let $(a_1, b_1, c_1) \leq_{OT} (a_2, b_2, c_2)$ and $(a_2, b_2, c_2) \leq_{OT} (a_1, b_1, c_1)$. Suppose that $A \neq B$. Then $a_1 \neq a_2$, $b_1 \neq b_2$ or $c_1 \neq c_2$. If $b_1 \neq b_2$ then either $A <_{OT}$ B or $B <_{OT}$ A but not both. So, $b_1 = b_2$. If $b_1 = b_2$ and $a_1 \neq a_2$ then, $A <_{OT} B$ or $B <_{OT} A$ but not both. Hence $b_1 = b_2$ and $a_1 = a_2$. If $b_1 = b_2$ and $a_1 = a_2$ and $c_1 \neq c_2$ then either $A <_{OT} B$ or $B <_{OT} A$ but not both. Thereby, $a_1 = a_2$, $b_1 = b_2$ and $c_1 = c_2$.

3. Transitivity: Let $(a_1, b_1, c_1) \leq_{OT} (a_2, b_2, c_2)$ and $(a_2, b_2, c_2) \leq_{OT} (a_3, b_3, c_3)$ is equivalent to

i) $b_1 < b_2 \leq b_3$ or $b_1 = b_2 < b_3$, or
ii) $b_1 = b_2 = b_3$ and $(a_1 < a_2 \leq a_3$ or $a_1 = a_2 < a_3)$, or
iii) $b_1 = b_2 = b_3$ and $a_1 = a_2 = a_3$ and $(c_1 \leq c_2 \leq c_3)$.

For the case i) we have to $b_1 < b_3$. For the other cases ii) and iii) it is analogous. Therefore, $(a_1, b_1, c_1) \leq_{OT} (a_3, b_3, c_3)$.

4. Totality: Let $A \neq B$, where $A = (a_1, b_1, c_1)$ and $B = (a_2, b_2, c_2)$. Then by Definition 3.1 and trichotomy property of real numbers we have $b_1 < b_2$ or $b_1 > b_2$ or $b_1 = b_2$. For the first two cases $(a_1, b_1, c_1) <_{OT} (a_2, b_2, c_2)$ or $(a_2, b_2, c_2) <_{OT} (a_1, b_1, c_1)$. Furthermore, if $b_1 = b_2$ then by Definition 3.1 and trichotomy property of real numbers $a_1 < a_2$ or $a_1 > a_2$ or $a_1 = a_2$. For the first two cases $(a_1, b_1, c_1) <_{OT} (a_2, b_2, c_2)$ or $(a_2, b_2, c_2) <_{OT} (a_1, b_1, c_1)$. But, if $a_1 = a_2$ then by Trichotomy on real numbers $c_1 < c_2$ or $c_1 > c_2$ or $c_1 = c_2$. Therefore, $A \leq_{OT} B$ or $B \leq_{OT} A$. In this way the Proposition is proven. □

Definition 3.2. *Let A be a TFN. Then,*

1. *A is OT-positive if* $A >_{OT} \widetilde{0}$.
2. *A is OT-negative if* $A <_{OT} \widetilde{0}$.

Proposition 3.2. *Let* (a, b, c) *be a TFN. Then,* (a, b, c) *is OT-positive if and only if* $b > 0$ *or (*$a = b = 0$ *and* $c > 0$*). Dually,* (a, b, c) *is OT-negative if and only if* $b \leq 0$ *and* $a < 0$.

Proof. Let (a, b, c) be a OT-positive TFN. Then, $0 < b$, or, $0 = b$ and $0 < a$ (contradiction), or, $0 = a$ and $0 = b$ and $0 < c$. The, converse is trivial. Moreover, let (a, b, c) be a OT-negative TFN. Then $b < 0$ (and therefore $a < 0$), or $b = 0$ and $a < 0$, or $a = b = 0$ and $c < 0$ (contradiction). So, $b \leq 0$ and $a < 0$. □

Corollary 3.1. *Let A be a TFN. Then, A is OT-positive if and only if* $-A$ *is OT-negative.*

Proof. Straightforward from Proposition 3.2. □

The following theorem proposes a property of sub-additive inverse.

Theorem 3.1. *Let A be a TFN. Then* $A - A \leq_{OT} \widetilde{0}$*. In addition,* $A - A = \widetilde{0}$ *if and only if A is crisp.*

Proof. Let $A = (a, b, c)$ be a TFN. By Proposition 2.2, $A - A = (a - c, b - b, c - a)$. Then $b - b = 0$ and $a - c \leq 0$. If $a - c = 0$ then $A - A = \tilde{0}$ and otherwise $A - A <_{OT} \widetilde{0}$. This completes the proof. In addition, $A - A = \widetilde{0}$ iff $(a - c, b - b, c - a) = (0, 0, 0)$ iff $a = c$ iff $a = b = c$ iff A is crisp. □

Theorem 3.2. *Let A and B two TFNs. If A and B are OT-positive then* $A + B$ *is also a OT-positive TFN.*

Proof. Let $A = (a_1, b_1, c_1)$ and $B = (a_2, b_2, c_2)$ two OT-positive TFNs. Then from Proposition 3.2 we have two cases. (1) $b_1 >$ or $b_2 > 0$ then $b_1 + b_2 > 0$. (2). $a_1 = b_1 = b_2 = a_2 = 0$ and $c_1 > 0$ and $c_2 > 0$ and therefore $a_1 + a_2 = b_1 + b_2 = 0$ and $c_1 + c_2 > 0$. Therefore, in both cases, $A + B >_{OT} \widetilde{0}$. □

Proposition 3.3. *Let A, B, C and D be TFNs. The following properties are satisfied*

1. *If* $A \leq_{OT} B$ *then* $A + C \leq_{OT} B + C$.
2. *If* $A \leq_{OT} B$ *and* $C \leq_{OT} D$ *then* $A + C \leq_{OT} B + D$.

Proof. Let $A = (a_1, b_1, c_1)$, $B = (a_2, b_2, c_2)$ and $C = (a_3, b_3, c_3)$ be TFNs. So, $A \leq_{OT} B$ if and only if the following assertions are verified:

i) $b_1 < b_2$, or
ii) $b_1 = b_2$ and $a_1 < a_2$, or
iii) $b_1 = b_2$ and $a_1 = a_2$ and $c_1 \leq c_2$.

Firstly we note that i) $b_1 + b_3 < b_2 + b_3$ or ii) $b_1 + b_3 = b_2 + b_3$ and $a_1 + a_3 < a_2 + a_3$, or $b_1 + b_3 = b_2 + b_3$, $a_1 + a_3 = a_1 + a_3$ and $c_1 + c_3 < c_2 + c_3$ then $A + C \leq_{OT} B + C$.

For 2. Straightforwardly from Definition 3.1 and assertion 1. This completes the proof. □

Proposition 3.4. *For each TFN A and B, there is a TFN X which is a solution of the inequality $A + X \geq_{OT} B$.*

Proof. Let A and B be TFNs. According with the Theorem 3.1 and Proposition 3.3, we have:

$$X \geq_{OT} (A - A) + X \geq_{OT} B - A.$$

Therefore $X \geq_{OT} B - A$. In this way the Proposition is proven. □

4. Selecting route to visit South American

In this section we collect distances between the capitals of South American countries on the following websites:

1. `https://www.distanciasentreciudades.com/`
2. `https://www.distancia.co/`
3. `https://www.geodatos.net/`
4. `https://es.distance.to/`

For example, between Caracas Venezuela to La Paz Bolivia where the distances are 3006.02, 2988, 3009, 3004.04 the number is $(2988, 3005.03, 3009)$. In this problem of the postman with 13 capital cities the possibilities of routes is very large number (6,227,020,800 routes) and it is possible to calculate it by the permutations considering 13 elements and only by adding the initial element at the end we would obtain the desired paths. Table 1 indicates the distances between the cities and the Figure 1 represents the locations on the continent. We have to consider that the names of the capitals are associated its acronym under the Code IATA[1].

Algorithm 1 For graph $G(V, E)$ find all paths and calculate total sum of paths. Finally select the shortest distance like Travelling Salesman Problem Algorithm.

for all w ∈ V **do**
    allpaths=findAllPaths(w,P)   ▷ P is the weight fuzzy of the path
    select-min(sort(allpaths))

[1]See `https://www.iata.org/en/publications/directories/code-search/`.

Algorithm 2 Find all paths from a given source to all nodes in a fuzzy weighted graph.

```
function FINDALLPATHS(startnode, FuzzP)
    init(addFuzz)
    v = startnode;
    e = {v, w}                                ▷ w some vertex different from the previous
    Add FuzzP[e] to addFuzz
    A = {v, w}                                ▷ A set of vertices visited
    if w is not visited then
        Add e to A;
    while (A! = V) do
        Find e = {v, w} | v ∈ A and w ∈ V − A
        Add FuzzP[e] to addFuzz
        Add w to A
    return A
```

Algorithm 2 determines all the routes that go through all the capitals of South America, starting from any one of them, to order them according to their cost. Algorithm 1 determines all routes starting from a specific capital. In addition, Table 2 shows the 10 shortest roads starting in Santiago de Chile. In our case, a Chilean who wants to go out traveling for the capitals of South America starting from Santiago, the algorithm, points out the shortest route.

Table 1. Triangular fuzzy numbers for distances between the capitals of the countries of South America

KM	ASU	BOG	BSB	BUE	CCS	GEO	LPB	LIM	MVD	PBM	POS	UIO	SCL
ASU		(3772, 3774.10, 3775.38)	(1460, 1463.84, 1712.77)	(1036, 1037.80, 1040)	(4082, 4105.06, 4108)	(3536.41, 3560.69, 3573)	(1462, 1465.06, 1466)	(2510, 2512.58, 2515)	(1069.33, 1075.89, 1078)	(3457, 3474.94, 3475.89)	(4000, 4020.63, 4025)	(3568, 3578.45, 3583)	(1547.80, 1550.48, 1555)
BOG	(3772, 3774.10, 3775.38)		(3280, 3663.50, 3671.16)	(4644, 4662.59, 4665)	(1019.86, 1026.89, 1028)	(1779.03, 1779.70, 1781)	(2424, 2438.58, 2442.27)	(1870, 1882.69, 1889.74)	(4759, 4776.36, 4781.46)	(2095.84, 2100.21, 2101)	(1533.53, 1538.40, 1541)	(728, 730.43, 737.12)	(4229, 4250.49, 4254)
BSB	(1460, 1463.84, 1712.77)	(3280, 3660, 3671.16)		(2333, 2346.20, 2677.81)	(3143.46, 3583, 3597.45)	(2290.46, 2730.77, 2748)	(2067.57, 2163, 2165.29)	(2987.61, 3168, 3171.08)	(2273, 2284.77, 2633.27)	(2074.20, 2524.50, 2538.49)	(2837.99, 3290.50, 3303.05)	(3452.82, 3776.50, 3781.41)	(3008, 3014.98, 3209.94)
BUE	(1036, 1037.80, 1040)	(4644, 4662.59, 4665)	(2333, 2342.20, 2677.81)		(5072, 5093.94, 5101)	(4571.11, 4594.45, 4610)	(2231, 2233.34, 2239)	(3131.73, 3132.28, 3141)	(203, 206.03, 210.55)	(4493, 4513.63, 4514)	(5022, 5043.85, 5052)	(4346, 4355.17, 4365)	(1129.08, 1135.19, 1140)
CCS	(4082, 4105.06, 4108)	(1019.86, 1026.89, 1028)	(3143.46, 3587, 3597.45)	(5072, 5093.94, 5101)		(1042.64, 1044.50, 1068.27)	(2988, 3005.03, 3009)	(2733, 2746.16, 2748)	(5149, 5169.24, 5178)	(1387.38, 1389.86, 1394)	(586.40, 588.50, 590.03)	(1751, 1753.57, 1754.65)	(4880, 4900.87, 4911)
GEO	(3536.41, 3560.69, 3573)	(1779.03, 1779.70, 1781)	(2290.46, 2730.77, 2756)	(4571.11, 4594.45, 4610)	(1042.64, 1044.50, 1068.27)		(2786.94, 2808.85, 2818)	(2941.43, 2954.22, 2960)	(4608.13, 4627.82, 4646)	(330.02, 346.68, 349)	(565, 566.12, 597.73)	(2388.50, 2391.67, 2392)	(4634.54, 4655.19, 4674)
LPB	(1462, 1465.06, 1466)	(2424, 2438.58, 2442.27)	(2067.57, 2163, 2165.29)	(2231, 2233.34, 2239)	(2988, 3005.03, 3009)	(2786.94, 2808.85, 2818)		(1076.49, 1076.99, 1078)	(2361, 2364.07, 2369)	(2857, 2867.27, 2867.84)	(3092, 3106.73, 3112)	(2130, 2136.97, 2141)	(1895, 1899.43, 1906)
LIM	(2510, 2512.58, 2515)	(1870, 1882.69, 1889.74)	(2987.61, 3169, 3171.08)	(3131.73, 3132.28, 3141)	(2733, 2746.16, 2748)	(2941.43, 2954.22, 2960)	(1076.49, 1076.99, 1078)		(3292.81, 3294.91, 3301)	(3127, 3132.38, 3133.96)	(3041, 3052.01, 3055)	(1317, 1325.29, 1327)	(2459, 2464.14, 2472)
MVD	(1069.33, 1075.89, 1078)	(4759, 4774.32, 4780)	(2273, 2280.77, 2633.27)	(203, 206.03, 210.55)	(5149, 5169.24, 5178)	(4608.13, 4627.82, 4646)	(2361, 2364.07, 2369)	(3292.81, 3294.91, 3301)		(4514, 4530.41, 4534.65)	(5075, 5093.62, 5103)	(4486, 4496.23, 4505)	(1339.13, 1339.64, 1341)
PMB	(3457, 3474.94, 3475.89)	(2095.84, 2100.21, 2101)	(2074.20, 2529, 2538.49)	(4493, 4513.26, 4514)	(1387.38, 1389.86, 1394)	(330.02, 346.68, 349)	(2857, 2867.27, 2867.84)	(3127, 3132.38, 3133.96)	(4514, 4530.41, 4534.65)		(878, 878.73, 883)	(2680.02, 2680.52, 2682)	(4649, 4664.37, 4670)
POS	(4000, 4020.63, 4025)	(1533.53, 1538.40, 1541)	(2837.99, 3294.53, 3304)	(5022, 5043.85, 5052)	(586.40, 588.50, 590.03)	(565, 566.12, 597.73)	(3092, 3106.73, 3112)	(3041, 3052.67, 3057.14)	(5075, 5093.62, 5103)	(878, 878.73, 883)		(2235, 2236.43, 2239)	(4978, 4997.12, 5009)
UIO	(3568, 3578.45, 3583)	(728, 730.43, 737.12)	(3452.82, 3778.50, 3781.41)	(4346, 4355.17, 4365)	(1751, 1753.57, 1754.65)	(2388.50, 2391.67, 2392)	(2130, 2136.97, 2141)	(1317, 1325.29, 1327)	(4486, 4496.23, 4505)	(2680.02, 2680.52, 2682)	(2235, 2236.43, 2239)		(3769, 3782.84, 3793)
SCL	(1547.80, 1550.48, 1555)	(4229, 4250.49, 4254)	(3008, 3012.48, 3209.94)	(1129.08, 1135.19, 1140)	(4880, 4900.87, 4911)	(4634.54, 4655.19, 4674)	(1895, 1899.43, 1906)	(2459, 2464.14, 2472)	(1339.13, 1339.64, 1341)	(4649, 4664.37, 4670)	(4978, 4997.12, 5009)	(3769, 3782.84, 3793)	

Figure 1. The shortest route from Santiago de Chile.

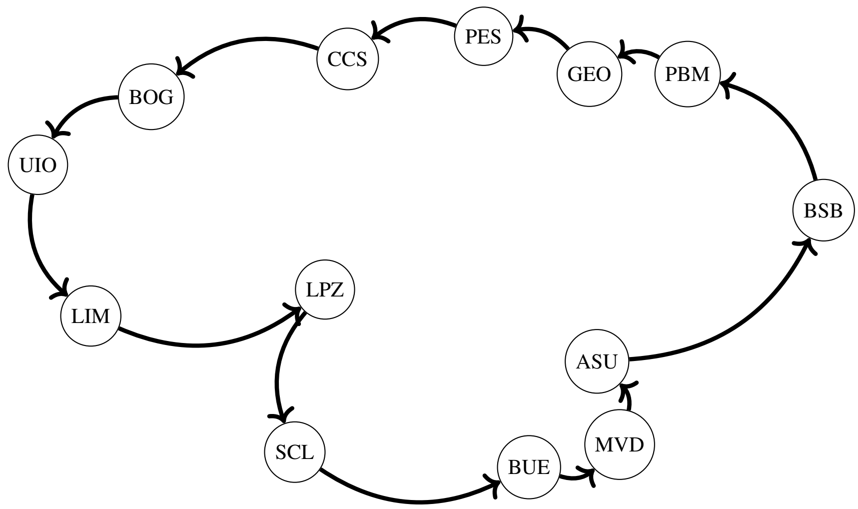

Table 2. The 10 shortest routes to the postman problem in South America.

$N°$	Route	Cost
1	SCL-LPB-LIM-UIO-BOG-CCS-POS-GEO-PBM-BSB-ASU-MVD-BUE-SCL	$(13464.3, 13970.6, 14292.7)$
2	SCL-BUE-MVD-ASU-BSB-PBM-GEO-POS-CCS-BOG-UIO-LIM-LPB-SCL	$(13464.4, 13972.3, 14292.7)$
3	SCL-LPB-LIM-UIO-BOG-CCS-POS-GEO-PBM-BSB-ASU-BUE-MVD-SCL	$(13632, 14133.5, 14455.7)$
4	SCL-MVD-BUE-ASU-BSB-PBM-GEO-POS-CCS-BOG-UIO-LIM-LPB-SCL	$(13641.1, 14138.7, 14455.7)$
5	SCL-BUE-MVD-ASU-BSB-GEO-PBM-POS-CCS-BOG-UIO-LIM-LPB-SCL	$(13993.6, 14491.2, 14787.5)$
6	SCL-MVD-BUE-ASU-BSB-GEO-PBM-POS-CCS-BOG-UIO-LIM-LPB-SCL	$(14170.4, 14657.6, 14950.5)$
7	SCL-BUE-MVD-ASU-LPB-LIM-UIO-BOG-CCS-POS-GEO-PBM-BSB-SCL	$(14770.34, 15277.54, 15349.84)$
8	SCL-BUE-MVD-ASU-BSB-PBM-GEO-POS-CCS-BOG-UIO-LPB-LIM-SCL	$(14843.4, 15350, 15672.7)$
9	SCL-MVD-BUE-ASU-LPB-LIM-UIO-BOG-CCS-POS-GEO-PBM-BSB-SCL	$(14947.04, 15443.84, 15512.84)$
10	SCL-BUE-MVD-ASU-LPB-LIM-UIO-BOG-CCS-POS-PBM-GEO-BSB-SCL	$(15299.54, 15796.34, 15844.64)$

5. Final remarks

We have presented a total order for triangular fuzzy numbers. Then we have shown properties that fulfill this order with the addition and subtraction operation, after studying these properties we can speak of a structure give for the triple $(\text{TFN}, +, \leq_{OT})$ that we could designate by *ordered Abelian quasi-group* or *ordered Abelian pseudo-group* (see [Clifford 1940, Levi 1942, Everett and Ulam 1945, Birkhoff 1987]) by the proposal of additive sub-inverse. In addition, we include an illustrative example for the Travelling Salesman Problem considering the capitals of those of South America.

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