An approach for consensual analysis on Typical Hesitant Fuzzy Sets via extended aggregations and fuzzy implications based on admissible orders

Monica Matzenauer[1], Advisor Renata Reiser [1], Coadvisor Helida Santos[2]

[1]Federal University of Pelotas (UFPel)
Pelotas, RS – Brazil

[2]Federal University of Rio Grande (FURG)
Rio Grande, RS – Brazil

{monica.matzenauer,reiser}@inf.ufpel.edu.br, helida@furg.br

Abstract. *Typical Hesitant Fuzzy Logic (THFL) is founded on the theory of Hesitant Fuzzy Sets, which consider as membership degrees the finite and non-empty subsets of the unit interval, called Typical Hesitant Fuzzy Elements (THFE). THFL provides the modelling for situations where there exists not only data uncertainty, but also indecision or hesitation among experts about the possible values for preferences regarding collections of objects. In order to reduce the information collapse for comparison and/or ranking of alternatives in the preference relationships, this thesis develops new ideas on THFL connectives, investigated under the scope of three admissible orders. In particular, properties of negations and aggregations are studied, as t-norms and OWA operators, with special interest in the axiomatic structures defining the implications and preserving their algebraic properties and representability. As the main contribution, we present a model that formally builds consensus measures on THFE through extended aggregation functions and fuzzy negation, using admissible orders for comparison and further, differentiating an analysis of consistency over preference matrices. Main theoretical results are submitted to multiple expert and mutiple criteria decision making problems.*
Keywords: *Typical hesitant fuzzy sets, Consensus measures, Fuzzy implications, Extended aggregation functions, Admissible orders.*

1. Introduction

Fuzzy Set Theory (FS), presented by [Zadeh 1965], has yielded several extensions over the years. Among the most relevant contributions in this research area, we highlight the following works:

- the extension known as Type-2 Fuzzy Sets (T2FS), introduced in [Zadeh 1971, Zadeh 1975], which considers the membership functions as FS on the unit interval $[0, 1]$;
- the Set-Valued Fuzzy Sets (SVFS), introduced by [Grattan-Guinness 1976] expressing the membership degrees as subsets of the unit interval $[0, 1]$;
- Atanassov's Intuitionistic Fuzzy Sets (IFS) in [Atanassov 1986], where the definition of a fuzzy set considers not only the membership function, but also its dual construction providing the non-membership degree; and

- Hesitant Fuzzy Sets (HFS), proposed by [Torra and Narukawa 2009] as another extension for fuzzy sets, in which the membership degree of an hesitant fuzzy set is also given as a subset of $[0, 1]$.

Note that despite the existence of different extensions in order to handle imprecise information, there are relationships among them as showed in [Bustince et al. 2016]. This work explored the inclusion relationships between some types of fuzzy sets and it also concluded that, in fact, the concepts of HFS and SVFS are equivalent. However, the results achieved in [Torra 2010] presented an explicit definition for the union and intersection operations on HFS, which was not the research focused in [Grattan-Guinness 1976].

Later, [Bedregal et al. 014a] noticed that in most applications, Typical Hesitant Fuzzy Elements (THFE) are used, i.e. finite and non-empty hesitant fuzzy degrees. Then, the notion of Typical Hesitant Fuzzy Logic (THFL) appears, which is based on Typical Hesitant Fuzzy Sets (THFS) conceived taking THFE as membership degrees.

Relevant research in decision making has been supported by the HFS theory, since it was introduced in 2010. See, for instance, the studies in [Zeng et al. 2020, Farhadinia et al. 2020, Rezaei and Rezaei 2020, Farhadinia and Xu 2020, Wang et al. 2021] considering the logical study of HFS. In particular, several weighted average (WA) and ordered weighted average (OWA)-like operators have been proposed to be used in multi-criteria and decision making (MCDM) problems dealing with multiple attributes and multiple specialists [Bedregal et al. 014a, Xia and Xu 2011, Zhu et al. 2012, Matzenauer et al. 2021a].

A frequent issue in the context of MCDM problems is that it is not always possible to find a consensus among a group of experts. So, it seems more appropriate to consider a set of possible values taking into account everyone's opinion. For instance, in order to provide a membership degree for an element of the universe, HFS can be useful to express this membership degree through a set of THFE, which will consider all the opinions given by the group of experts.

However, some research questions arise from this setting:

(i) How much do these elements agree with each other?
(ii) Is it possible to combine these elements into a single output?
(iii) Is the result reliable and does it reflect the opinions provided by the group?

The consolidated research on consensus measures provide relevant results contributing to answer all these questions, which have been applied in different contexts. In the current literature, we can find works on fields like consensual processes [Unzu and Vorsatz 2011], consensual measures and aggregations [Beliakov et al. 2014], majority decisions [Garcí a Lapresta and Llamazares 2010] and preference intensities group decision and negotiation [García-Lapresta and Llamazares 2001, Llamazares et al. 2013].

Apart from other results, we provide a general idea of up to what extent the expert inputs agree with one another based on our approach using the theory of THFS. In our proposal, we present a model that formally constructs consensus measures by means of aggregations functions, fuzzy implication-like functions and fuzzy negations, using admissible orders to compare the THFE, and also providing an analysis of consistency on them. Our theoretical results are applied into a problem of decision making with multi-criteria illustrating our methodology to achieve consensus in a group of experts working with typical hesitant fuzzy sets.

2. Main contributions

As the main contribution, the thesis introduces a model to provide semantic interpretation for consensus setting on Typical Hesitant Fuzzy Sets, namely the $CC_{\mathbb{H}}$-Model: Consensus Measures on Typical Hesitant Fuzzy Sets (THFS) based on Extended Aggregation and Implication Operators.

The main properties proposed in the literature of consensus measures are studied here from the setting of inputs on the class of THFS defined over $\mathbb{H}$, which is the set of all finite and non-empty subsets of the unit interval $[0, 1]$, mainly according to the approach in [Beliakov et al. 2014]. However, the studies are restricted to the fact that the agreement among evaluations has the same relevance.

Going beyond, among several partial orders defined over $\mathbb{H}$, the proposal represents an extension that considers an admissible partial order $\leq_A$ on $\mathbb{H}$; based on an extended aggregation operator A, as a binary relation promoting comparison even between THFE with different cardinalities.

Main contributions achieved in the thesis are listed below:

(i) Starting with fuzzy extensions of consensus measures from U to $\mathbb{H}$ and taking into account so many distinct partial orders for THFS, this work introduces a new admissible order based on a hesitant aggregation function $\mathcal{A}$, refining not only the restricted consensual order $<_{RH}$ but many other ones, providing a comparison between HFS which do not have the same cardinalities;

(ii) This research considers the concepts of admissible orders obtained from hesitant aggregation functions and fuzzy negations, providing methods to generate comparisons (ordering) of typical hesitant fuzzy elements [Bustince et al. 2013, Miguel et al. 2016, Lima 2019];

(iii) Extension of the main concepts of fuzzy connectives (fuzzy negations, aggregation functions, t-norms and t-conorms, fuzzy implications) are considered, discussing their properties regarding admissible orders [Bustince et al. 2020, Matzenauer et al. 2021a];

(iv) The work also considers a formal definition of consensus measures extending this concept to the typical hesitant fuzzy sets. Actually, various applications can be found in the literature concerning consensus measures [Li et al. 2018, Rodríguez et al. 2018], and also an attempt of a formal mathematical definition was given in [Beliakov et al. 2014].

(v) This study explores methodologies based on the $CC_{\mathbb{H}}$-Model, a construction of consensus measures through different implications, exploring main properties of fuzzy implications which are required, regarding admissible/total orders on $\mathbb{H}$.

3. Objectives

The main objective of the thesis is to extend Beliakov's work by applying consensus measures to typical hesitant fuzzy sets based on aggregation functions and admissible orders.

The specific objectives are described as follows:

- Contribute for the study of hesitant fuzzy sets and admissible linear orders, considering their relevance in multi-valued fuzzy sets by allowing comparison and ordering relations;
- Collaborate to the study of hesitant fuzzy aggregators, exploring the main properties, analysing their relevance to consensus measurement methodologies and practical applications;
- Study fuzzy consensus measures and provide the application of related methodology in the decision making based on multi-criteria and -attribute from many specialists;
- Introduce the study of main classes of hesitant fuzzy implications, exploring the main properties and constructors of duality and conjugation;
- Propose the $CC_{\mathbb{H}}$-Model, based on an axiomatic definition of consensus measures consistent with studies of admissible linear orders in typical hesitant fuzzy sets.

4. Final Considerations

In the thesis, new ideas in THFE are investigated and developed under the scope of an arbitrary order, allowing the possibility of comparisons of THFE with different cardinalities. A class of admissible linear $\langle \mathbb{H}, \preceq_A^f \rangle$-orders is also presented, when A is an increasing aggregation function and f satisfies the injective-cardinality property. This admissible $\langle \mathbb{H}, \preceq_A^f \rangle$-orders is a family of total orders that refine the partial $\langle \mathbb{H}, \leq_{RH} \rangle$ -order. The theorem presented below, can be seen in with more details in the thesis and in [Matzenauer et al. 2021b].

Theorem 4.1 *Let $A^* : \mathbb{H} \to [0,1]$ be a function such that A^* is increasing w.r.t. $\leq_{RH}$, $A^*(\mathbf{0}_{\mathbb{H}}) = 0$ and $A^*(\mathbf{1}_{\mathbb{H}}) = 1$ and $f^* : \mathbb{H} \to \mathbb{R}$ be a function such that the property:*

$IC : f^*(X) = f^*(Y) \Rightarrow \#X = \#Y$ *(injective-cardinality property)*

is satisfied. The relation $\preceq_{A^*}^{f^*}$ on $\mathbb{H}$ defined by

$$X \preceq_{A*}^{f^*} Y \Leftrightarrow \begin{cases} X = Y, \text{ or} \\ A^*(X) < A^*(Y), \text{ or} \\ A^*(X) = A^*(Y) \text{ and } f^*(X) < f^*(Y), \end{cases} \tag{1}$$

is a total admissible order on $\mathbb{H}$ whenever, for each $n \in \mathbb{N}^+$, $A_n^* = A^* \upharpoonright \mathbb{H}^{(n)}$ is injective.

Two other admissible linear orders are also considered: $\langle \mathbb{H}, \preceq_{Lex1} \rangle$- and $\langle \mathbb{H}, \preceq_{Lex2} \rangle$-orders, that allowed us to introduce a formal definition of hesitant fuzzy operators, such as $\langle \mathbb{H}, \preceq \rangle$-aggregation functions and $\langle \mathbb{H}, \preceq \rangle$-negations, including their respective important properties. Emphasizing as main results, we have the generation of $\langle \mathbb{H}, \preceq \rangle$-negations from fuzzy negations, also presenting some interesting examples.

The contextual theoretical research on the properties of $\langle \mathbb{H}, \preceq \rangle$-implications is related to the monotonicity analysis, which is restricted to the first place antitonicity and second place isotonicity, but also including the identity and exchange principles, the left and right boundary conditions, the contrapositive symmetry and ordering properties. Some additional examples are also presented, mainly related to natural negations obtained from $\langle \mathbb{H}, \preceq \rangle$-implication functions.

Based on such classes of $\langle \mathbb{H}, \preceq \rangle$-orders, expressions for the main examples of aggregation functions and fuzzy implications are presented. Furthermore, the representability of such operators is obtained from generation of $\langle \mathbb{H}, \preceq \rangle$-implications as an order-preserving structure of main implication properties. Besides, is introduced a formal definition for the representability of those negations, by constructing a method to obtain $\langle \mathbb{H}, \preceq \rangle$-implications from $\langle \mathbb{H}, \preceq \rangle$-aggregations.

Another relevant contribution illustrating our theoretical results is an algorithmic solution for an ME-MCDC problem, which used $\langle \mathbb{H}, \preceq \rangle$-operators and took into account the selection of a CIM-software. By applying the Łukasiewicz implication, the example reported an ME-MCDM problem in a CIM-application, which could be analysed from three distinct comparisons based on $\langle \mathbb{H}, \preceq \rangle$-operators.

The work also presents improved consensus-based procedures based on admissible $\langle \mathbb{H}, \preceq \rangle$-orders, handling Multi Expert-Multi Criteria Decision Making (ME-MCDM) problems and using consistent $\langle \mathbb{H}, \preceq \rangle$-preference relations (HFPR). At the first level, the consistence analysis considers the weak transitivity and ordinal consistency properties in $\langle \mathbb{H}, \preceq \rangle$-orders, also extending the notion of (restricted) $\max$-$\max$ and $\min$-$\max$ transitivity. Subject to such results on consistency analysis, normalised additive hesitant fuzzy preference relations introduce two strategies to obtain a consensus-based model.

Then, we formally defined the generalised notion of consensus measures from $([0, 1], \leq)$ to a bounded poset $\mathbb{H} = \langle \mathbb{H}, \preceq \rangle$, also studying the corresponding extensions of aggregations, implications and fuzzy negations. As one of the main contribution, the $\mathcal{CC}_{\mathcal{AI}}$- and $\mathcal{CC}_{\min,\mathcal{I}}$-consensus models are presented as new methodologies of consensus preserving main properties in the context of Typical Hesitant Fuzzy Sets, by exploring properties of admissible $\langle \mathbb{H}, \preceq \rangle$-aggregation and admissible $\langle \mathbb{H}, \preceq \rangle$- implications.

This first consensus measures method $\mathcal{CC}_{\mathcal{AI}}$ is based on a typical hesitant extended aggregation function $\mathcal{A}$ and an $\langle \mathbb{H}, \preceq \rangle$-implication $\mathcal{I}$. The theorem presented below, can be seen with more details in the thesis and in [Matzenauer et al. 2021b].

Theorem 4.2 *Let $\mathcal{A}$ be an extended $\langle \mathbb{H}, \preceq \rangle$-aggregation function satisfying some aggregations properties and $\mathcal{I}$ be an $\langle \mathbb{H}, \preceq \rangle$-implication verifying some implications properties. Then the operator $\mathcal{CC}_{\mathcal{AI}} \colon \bigcup_{n=2}^{\infty} \mathbb{H}^n \to \mathbb{H}$ given by*

$$\mathcal{CC}_{\mathcal{AI}}(X_1, \ldots, X_n) \; = \; \mathcal{A}^n_{i,j=1, i\neq j}(\mathcal{I}(X_i, X_j)), \tag{2}$$

is an $\langle \mathbb{H}, \preceq, \mathbf{0}_{\mathbb{H}}, \mathbf{1}_{\mathbb{H}} \rangle$-valued consensus measure on $\mathbb{H}$,

This second method, $\mathcal{CC}_{\min,\mathcal{I}}$-consensus model, considers the minimum typical hesitant aggregation ($\mathcal{T}_{\min} = \min$) and an $\langle \mathbb{H}, \preceq \rangle$-implication $\mathcal{I}$, and is presented with more details in the thesis and in [Matzenauer et al. 2021b].

Theorem 4.3 *Let $\mathcal{A}$ be an extended $\langle \mathbb{H}, \preceq \rangle$-aggregation function satisfying some aggregations properties and $\mathcal{I}$ be an $\langle \mathbb{H}, \preceq \rangle$-implication verifying some implications properties. The operator $\mathcal{CC}_{\min,I} \colon \bigcup_{n=2}^{\infty} \mathbb{H}^n \to \mathbb{H}$ given by*

$$\mathcal{CC}_{\min,I}(X_1, \ldots, X_n) \; = \begin{cases} \mathcal{I}(X_1, X_2) \wedge \mathcal{I}(X_2, X_1), & \textit{if } n = 2 \\ \mathcal{CC}_{\min,I}(X_1, \mathcal{CC}_{\min,I}(X_2, \ldots, X_n)), & \textit{if } n > 2, \end{cases} \tag{3}$$

is an $\langle \mathbb{H}, \preceq \rangle$-valued consensus measure on $\mathbb{H}$.

5. Future works

Ongoing works are focusing on other classes of implications, such as (S, N)-implications [Zanotelli et al. 2020] and including the residuation principle related to R-implications and the left-continuity of t-norms, in the context of admissible $\langle \mathbb{H}, \preceq \rangle$-orders. And, in order to show the advantage of the proposed method, further work extends case studies in cloud computing [Schneider et al. 2020], for hesitant fuzzy environments, based on the theoretical results achieved in this step of the research on admissible linear orders.

We also intend to explore new group of strategies to obtain consensus measures, mainly connected to the class of operators, satisfying commutative, nondecreasing aggregations with $\mathbf{1}_{\mathbb{H}}$-annihilator.

6. Publications

This section reports the publication of the main results connected with the thesis.

1. MATZENAUER, M. L.; SANTOS, H.; BEDREGAL, B.; BUSTINCE, H.; REISER, R.. On Admissible Total Orders for Typical Hesitant Fuzzy Consensus Measures. in: International Journal of Intelligent Systems (IJIS, 2021), p. 1-23, 2021.
2. MATZENAUER, M. L.; REISER, R.; SANTOS, H.; BEDREGAL, B.; BUSTINCE,H.. Strategies on Admissible Total Orders Over Typical Hesitant Fuzzy Implications Applied to Decision Making Problems. In: International Journal of Intelligent Systems. (IJIS, 2021), p. 1-50, 2021.
3. MATZENAUER, M. L.; REISER, R.; SANTOS, H.; PINHEIRO, J.; BEDREGAL, B.. An Initial Study on Typical Hesitant (T,N)-Implication Functions. In: 18th International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems. CCIS 1238, p. 747-760, 2020.
4. MATZENAUER, M. L.; REISER, R.; SANTOS, H.; BEDREGAL, B.. Typical Hesitant Fuzzy Sets: Evaluating Strategies in GDM Applying Consensus Measures. In: Conference of the International Fuzzy Systems Association and the European Society for Fuzzy Logic and Technology. (EUSFLAT 2019), 2019., 2019/08. Anais. Atlantis Press, 2019/08.
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