n-Dimensional Fuzzy Implications: Analytical, Algebraic and Applicational Approaches

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Abstract. *The study of n-dimensional fuzzy logic contributes to overcome the insufficiency of traditional FL in modeling imperfect and imprecise information coming from different experts. Based on representability, we extend results from fuzzy connectives to n-dimensional approach. This research on n-dimensional fuzzy implications (n-DI) pass through the next steps: (i) analytical studies;(ii) algebraic aspects;(iii) n-dimensional approach of fuzzy implication classes represented by fuzzy connectives as (S, N)-implications and QL-implications; (iv) studies of n-dimensional R-implications (n-DRI); (v) constructive method obtaining n-DRI based on n-dimensional aggregation operators and (vi) an introductory study considering an n-DI in modeling approximate reasoning.*

1. Introduction

The n-dimensional interval studies provide a new possibility to model not only the imprecision in the calculations, but also to describe the (ordered and possible repeated) uncertainty opinion provided by many specialists related to their preference relations faced to multi-criteria data and decision making systems.

In order to contribute with the theoretical research on n-dimensional intervals, this thesis aims to study of n-dimensional fuzzy sets (n-FS), not only as an extension of fuzzy sets, but also investigating their properties, exploring main classes of n-dimensional aggregations (n-DA) and also reporting two applications.

More specifically, this research proposes a theoretical study of n-DI considering the following approaches:
(i) the analytical study, defining n-DI, presenting the most desirable properties;
(ii) the algebraic aspects related to the left and right continuity n-dimensional t-norms and properties of natural n-dimensional fuzzy negations;
(iii) generation of n-DI by analysing their representability from fuzzy implications, also investigating the conjugation operation by action of n-dimensional automorphisms;
(iv) the n-DI classes explicitly/implicitly represented by fuzzy connectives, such as

(S, N)-implications and QL-implications;
(v) the study of n-DRI, analyzing the Residuation Principle and characterizing n-DRI based on continuous n-dimensional Triangular Norms (n-DT);
(vi) providing constructive methods to obtain n-DRI via n-dimensional aggregations.

Thus, these potential results illustrate applications of n-DRI in two fields:
(i) in solving problems of Multiple Criteria and Decision Making (MCDM), considering multiple alternatives in the selection of Computer-Integrated Manufacturing (CIM) software; and,
(ii) in modelling the approximate reasoning (AR) of fuzzy inference schemes, considering n-dimensional extension of intervals, fuzzy statements and IF-THEN rules.

1.1. Main Motivations

This large field of theoretical research and technological applications related to n-dimensional fuzzy set theory motivates the extensive studies developed in this thesis. In particular, it is reported in [Zanotelli 2020] that the class of fuzzy implications plays an important role in inference fuzzy systems, since they are applied in fuzzy control, imprecision analysis from natural language and soft-computing techniques. They provide the mathematical basis for the generation of powerful techniques for AR involving uncertain, vague and nebulous processes. Such theoretical results underlie applications in areas as pattern recognition, image processing, data mining and mathematical morphology, modeling the relevance of frequency in repetitive data information.

In addition, the extension of fuzzy implication arises in many situations in various branches of computer science and technological development, see, e.g., controlling a mobile robot; providing prediction of the survival time of myeloma patients in hybrid systems for genetic algorithms; modelling expert system for shopping via web; developing systems for analysis and estimation of the survival time of wireless sensor networks.

As another relevant motivation, we consider to consolidate projects connected to Laboratory of Ubiquitous and Parallel Systems (LUPS), Formal Methods and Mathematical Fundamentals of Computer Science (MFFMCC), LoLiTA (UFRN/Brazil) and GIARA (Navarra/Spain), research groups in multi-valued fuzzy logics showing the conditions under which the main properties of fuzzy implications are preserved by representable n-DI.

1.2. Contextual approach of the research proposal and related works

In 1983, the Atanassov's intuitionistic fuzzy sets [Atanassov 1986] (AFS) were defined as a special type of Type-2 Fuzzy Set (T2FSs), in such a way that each element is associated with a pair of real numbers, representing the membership and non-membership degrees, which is not necessarily defined by the complementary relationship.

Basic ideas of interval-valued fuzzy sets (IVFS) were simultaneously defined by Zadeh [Zadeh 1975] and Sambuc [Sambuc 1975]. Relevant works in IVFS were preliminarily studied by Mizumoto and Tanaka [Mizumoto and Tanaka 1976] also considering Dubois and Prade's [Dubois and Prade 2005] and [Deschrijver and Kerre 2005] research. And, the notion of interval-valued Atanassov intuitionistic fuzzy sets (IVAFS) is introduces in [Atanassov and Gargov 1998], formalizing AFS in the field of T2FSs.

Vicenç Torra proposes hesitant fuzzy sets (HFS) as an extension of the fuzzy sets in [Torra 2010] with the goal of considering fuzzy sets where the membership degree can be expressed as a set of values, enabling the description of distinct opinions of several experts. Later, in [Bedregal et al. 2014] the notion of typical hesitant fuzzy sets (THFS) considers only finite and nonempty hesitant fuzzy values.

In 2010, n-dimensional fuzzy sets (n-DFS) were conceived by [Shang et al. 2010] as an extension of fuzzy sets, including IVFS and IVAFS. As the main difference related to HFS, n-DFS allow you to explore repetition of ordered elements on the membership degrees. So, as the main idea, n-DFS consider several uncertainty levels in the memberships degrees to model different membership degrees which are obtained as a consensus to unify or even aggregate these values [Bedregal et al. 2011].

See Table 1 summarizing the main characteristics of the selected research papers on $L_n(U)$, reporting techniques, main contributions and the applied approach areas.

1.3. Methodological structure of the doctoral thesis

This thesis is organized in eight Chapters, starting from preliminary aspects of fuzzy logic and evolving to more specific ones in order to develop the research.

In Chapter 2, the contributions of the type-n fuzzy sets to the systems development area are treated, a comparative and hierarchical analysis of the fuzzy sets is made showing the different interpretations of information and a literature review reporting relevant studies on the n-dimensional intervals.

The fundamental concepts of fuzzy logic is shown in Chapter 3, starting with automorphisms, connectives in fuzzy logic such as negations, triangular norms and conorms and implications with the main classes.

Fundamental concepts of n-dimensional fuzzy logic (n-DFL), such as order relations, the main properties of continuity of functions, automorphism, and an extension of the fuzzy sets are reported in Chapter 4.

In sequence, in Chapter 5, the n-dimensional fuzzy implications are reported, showing the main properties as well as representability. And finally, the expansion of the classes (S, N)-implication and QL-implication.

In Chapter 6, it deals with the extension of residual fuzzy implication, showing its main properties and also the action of automorphism in n-DRI. We present the characterization of n-DRI. Still how to obtain from n-dimensional aggregation operators an n-DRI and finally the extension of a fuzzy application to n-dimensional fuzzy.

Focusing on the R-implications, the discussions extending the residuation property based on Moore-metric, continuity and monotonicity properties with respected to the usual partial orders on $L_n(U)$ are considered in Chapter 7. Extending results from interval-valued fuzzy set theory to n-dimensional simplex, the class of n-DRI is constructed based on n-DA aggregation operators, given as the minimum operator and left-continuous t-norms. Some topics refer to the residuation property, conjugate operators, characterization of n-DRI, application of of n-DA to obtain n-DRI and main properties preserved by such methodology. Some basic concepts of AR are extended to the n-dimensional fuzzy approach, showing that n-dimensional fuzzy deduction rules generalize fuzzy rules of classical logic.

Thus, in conclusion, final considerations are presented, highlighting the original-

Table 1. Related papers on n-dimensional fuzzy sets

	Technique	Contribution	Class
1	Study on n-Dimensional R-implications [Zanotelli et al. 2021]	Study of the properties that characterize the class of R-**implications on** $L_n(U)$.	TR[1] DMP[2]
2	Conjuntos Fuzzy Multidimensionais [Lima 2019]	Presents the concept of **multidimensional fuzzy sets** as a generalization of n-DFS.	TR
3	n-Dimensional Interval Uninorms [Mezzomo et al. 2019]	Presents conjugate based on n-**dimensional interval uninorms** preserving their main properties.	TR
4	n-Dimensional Intervals and Fuzzy S-implications [Zanotelli et al. 2020]	Study main properties characterizing the class of S-**implications on** $L_n(U)$.	TR
5	Towards the study of main properties of n-Dimensional QL-implicators [Zanotelli et al. 2018]	Introduces the concept of n-**dimensional** QL-**implicators** considering duality and conjugation operators.	TR
6	Equilibrium Point of Representable Moore Continuous n-Dimensional Interval Fuzzy Negations [Mezzomo et al. 2018a]	Studies the class of representable (Moore-continuous) n-**dimensional interval fuzzy negations**.	TR
7	Moore Continuous n-Dimensional Interval Fuzzy Negations [Mezzomo et al. 2018b]	Characterizes the notion of (continuous) n-dimensional interval Moore-metric and n-**dimensional interval fuzzy negations**.	TR
8	n-Dimensional Fuzzy Negations [Bedregal et al. 2018]	Investigates an extension of n-representable fuzzy **negations on** $L_n(U)$ which are continuous and monotone.	MEDM[3]
9	Natural n-dimensional fuzzy negations for n-dimensional t-norms and t-conorms [Mezzomo et al. 2017]	Considers n-dimensional fuzzy negations studying n-**dimensional triangular norms and triangular conorms**.	TR
10	An algorithm for group decision making using n-dimensional fuzzy sets, admissible orders and OWA operators [De Miguel et al. 2017]	Introduces the concept of **admissible order for** n-**dimensional fuzzy set** as well as a construction method for these orders.	GDMP[4]
11	On n-dimensional strict fuzzy negations [Mezzomo et al. 2016]	Investigate the class of representable n-**dimensional strict fuzzy negations**.	TR
12	Weighted Average Operators Generated by n-dimensional Overlaps and an Application in Decision Making [Silva et al. 2015]	Provides a way to generate a class of **weighted average operator from** n-**dimensional** overlap functions and aggregation functions.	MAGDMP[5]
13	A class of fuzzy multisets with a fixed number of memberships [Bedregal et al. 2012]	Define a generalization of Atanassov's operators for n-dimensional fuzzy values (called n-dimensional intervals).	DMP
14	A Characterization Theorem for t-Representable n-Dimensional Triangular Norms [Bedregal et al. 2011]	Generalize the notion of t-representability for n-**dimensional t-norms** and provide a characterization theorem for that class of n-dimensional t-norms.	TR
15	The n-dimensional fuzzy sets and Zadeh fuzzy sets based on the finite valued fuzzy sets [Shang et al. 2010]	Defines cut sets on n-DFS studying the **decomposition and representation theorems**.	TR

[1]Theoretical Research [2]Decision Making Problem [3]Multi-Expert Decision Making
[4]Group Decision Making Problems [5]Multiple Attribute Group Decision Making Problems

ity and relevance of the thesis, the main contributions, the publications made during the doctorate course and the possibilities for future work to be developed.

2. n-Dimensional Fuzzy Logic: Fundamental Concepts

An n-dimensional fuzzy set considers tuples of size n whose components are valued in $U = [0; 1]$ and ordered in increasing form, called n-dimensional intervals. Generally, these sets contribute in modeling situations involving decision-making where, each n-dimensional interval represents the opinion of n specialists on the degree to which an alternative meets a given criterion or attribute for this problem. The membership values of n-DFS are n-tuples of real numbers in $U = [0, 1]$, called n-dimensional intervals, which are ordered in an increasing order. Formally, let $\chi \neq \emptyset$ and $\mathbb{N}_n = \{1, 2, \ldots, n\}$. By [Shang et al. 2010], an n-dimensional fuzzy set $A_{L_n(U)}$ is expressed as

$$A_{L_n(U)} = \{(\mathrm{x}, \mu_{A_{L_n(U)}}(\mathrm{x}))\colon \forall \mathrm{x} \in \chi\}, \tag{1}$$

where $\forall \mathrm{x} \in \chi$, $\mu_{A_{L_n(U)}}(\mathrm{x}) = (x_1, x_2, \ldots, x_n)$ such that $x_1 \leq x_2 \leq \ldots \leq x_n$. So, an n-dimensional fuzzy set B over χ can also be given as $B_{L_n(U)} = \{(\mathrm{x}, \mu_{B1}(\mathrm{x}), \ldots, \mu_{Bn}(\mathrm{x})) : \mathrm{x} \in \chi\}$, where all membership functions of B, denoted as $\mu_{Bi} : \chi \to U$, $\forall i \in \mathbb{N}_n$ verifies the condition $\mu_{B1}(\mathrm{x}) \leq \ldots \leq \mu_{Bn}(\mathrm{x})$, $\forall \mathrm{x} \in \chi$. The n-dimensional upper simplex is given as $L_n(U) = \{\mathbf{x} = (x_1, \ldots, x_n) \in U^n : x_1 \leq \ldots \leq x_n\}$, and an element $\mathbf{x} \in L_n(U)$ is called n-dimensional interval.

2.1. Improving the residuum analysis on $L_n(U)$

The following results show the necessary and sufficient conditions under which the pair of functions $(\mathcal{I}_\mathcal{T}, \mathcal{T})$ verifies the residuation property on $\mathcal{L}_n(U)$.

Theorem 1 *Let $\mathcal{T}$ be an n-DT. The following statements are equivalent:*
***1**. $\mathcal{T}$ safisties the left-continuity property: $\mathcal{LC} : \lim_{n\to\infty} \mathcal{T}(\mathbf{x}, \mathbf{y}_n) = \mathcal{T}(\mathbf{x}, \lim_{n\to\infty} \mathbf{y}_n)$;*
***2**. $(\mathcal{T}, \mathcal{I}_\mathcal{T})$ is an adjoint pair, verifying the residuation property: $\mathcal{RP} : \mathcal{T}(\mathbf{x}, \mathbf{z}) \leq \mathbf{y} \Leftrightarrow \mathcal{I}_\mathcal{T}(\mathbf{x}, \mathbf{y}) \geq \mathbf{z}$; **3**. $\mathcal{I}_\mathcal{T}(\mathbf{x}, \mathbf{y}) = \max\{\mathbf{z} \in L_n(U)\colon \mathcal{T}(\mathbf{x}, \mathbf{z}) \leq \mathbf{y}\}$.*

The characterization of n-DRI is formalized from left-continuous n-DT considering the method of obtaining n-DT from n-dimensional fuzzy implications by a residuation principle. Since each $\mathcal{I} \in \mathcal{I}(L_n(U))$ verifies the right boundary condition, meaning that $\mathcal{I}(\mathbf{x}, /1/) = /1/$, the function $\mathcal{T}_\mathcal{I} : (L_n(U))^2 \to L_n(U)$,

$$\mathcal{T}_\mathcal{I}(\mathbf{x}, \mathbf{y}) = inf\{\mathbf{t} \in L_n(U)\colon \mathcal{I}(\mathbf{x}, \mathbf{t}) \geq \mathbf{y}\}, \tag{2}$$

is a well-defined function on $L_n(U)$.

Proposition 1 *For $\mathcal{I} \in \mathcal{I}(L_n(U))$ the following statements are equivalent:*
1. *$\mathcal{I}$ is right-continuous w.r.t. the second variable.*
2. *$(\mathcal{T}_\mathcal{I}, \mathcal{I})$ is an adjoint pair.*
3. *The infimum in Eq.(2) is the minimum, i.e., $\mathcal{T}_\mathcal{I}(\mathbf{x}, \mathbf{y}) = \min\{\mathbf{t} \in L_n(U)\colon \mathcal{I}(\mathbf{x}, \mathbf{t}) \geq \mathbf{y}\}$, where the right side exists for all $\mathbf{x}, \mathbf{y} \in L_n(U)$.*

Theorem 2 *If a function $\mathcal{I} : (L_n(U))^2 \to L_n(U)$ satisfies $\mathcal{I}2$, $\mathcal{I}5$, $\mathcal{I}13$ and verifies the right-continuity w.r.t. the second variable, then $\mathcal{T}_\mathcal{I}$ is a left-continuous n-DT. Moreover $\mathcal{I} = \mathcal{I}_{\mathcal{T}_\mathcal{I}}$, meaning that $\mathcal{I}(\mathbf{x}, \mathbf{y}) = \max\{\mathbf{t} \in L_n(U)\colon \mathcal{T}_\mathcal{I}(\mathbf{x}, \mathbf{t}) \leq \mathbf{y}\}$.*

And, in the next results, n-DRI can be obtained from n-DA operators.

Proposition 2 *Let* $T_1, \ldots, T_n : U^2 \to U$ *be left-continuous t-norms such that* $T_1 \leq \ldots \leq T_n$*. Then, for all* $\mathbf{x} = (x_1, ..., x_n),\ \mathbf{y} = (y_1, ..., y_n) \in L_n(U)$ *the function* $\mathcal{T}_{T_1,\ldots,T_n} : (L_n(U))^2 \to L_n(U)$ *given as*

$$\mathcal{T}_{T_1 \ldots T_n}(\mathbf{x}, \mathbf{y}) = \left(\min_{i=1}^{n} T_1(x_{n-i+1}, y_i), \min_{i=2}^{n} T_2(x_{n-i+2}, y_i), \ldots, T_n(x_n, y_n) \right), \qquad (3)$$

verifies the t-subnorm conditions and $\mathcal{LC}$ *and* $\mathcal{RP}$ *properties on* $L_n(U)$.

The characterization of an $\mathcal{I}_{\mathcal{T}_{T_1,\ldots,T_n}}$ operator is given in the following theorem.

Theorem 3 *Let* $T_1, \ldots, T_n : U^2 \to U$ *be left-continuous t-norms on* U *such that* $T_1 \leq \ldots \leq T_n$ *and* $I_1, \ldots, I_n$ *be their corresponding residual implications. Then the operator* $\mathcal{I}_{I_1,\ldots,I_n} = \mathcal{I}_{\mathcal{T}_{T_1,\ldots,T_n}}$ *where* $\mathcal{I}_{I_1,\ldots,I_n} : (L_n(U))^2 \to L_n(U)$ *defined as*

$$\mathcal{I}_{I_1,\ldots,I_n}(\mathbf{x}, \mathbf{y}) = \left(\min_{i=1}^{n} I_i(x_i, y_i), \min_{i=2}^{n} I_i(x_i, y_i), \ldots, I_n(x_n, y_n) \right) = \left(\min_{i=k}^{n} I_i(x_i, y_i) \right)_{k \in \mathbb{N}_n^*}.$$

Corollary 1 *Let* $T_1, \ldots, T_n : U^2 \to U$ *be left-continuous t-norms on* U *such that* $T_1 \leq \ldots \leq T_n$ *and* $I_1, \ldots, I_n$ *be their corresponding residual implications. Then* $(\mathcal{I}_{I_1,\ldots,I_n}, \mathcal{T}_{T_1 \ldots T_n})$ *is an adjoint pair on* $L_n(U)$.

See now, in the following, a constructive method of an n-DI.

Proposition 3 *Let* $I_1, \ldots, I_n : U^2 \to U$ *be implications. Then, function* $\mathcal{I}_{I_1,\ldots,I_n} : (L_n(U))^n \to L_n(U)$ *is an* n*-DI:*

$$\mathcal{I}_{I_1,\ldots,I_n}(\mathbf{x}, \mathbf{y}) = \left(\min_{i=1}^{n} I_i(x_i, y_i), \min_{i=2}^{n} I_i(x_i, y_i), \ldots, I_n(x_n, y_n) \right), \qquad (4)$$

3. Relevance and originality of the thesis

The study of n-dimensional intervals provide a new possibility to model not only the imprecision in the calculations, but also to describe the (possible repeated) uncertainty opinion provided by many specialists related to their preference relations faced to multi-criteria data. In these contexts, n-DFL approach contributed for the development of applications based on MEDM problems.

Such ordered structures are extensions of FS as old as HFS but less explored, in theoretical and applied research. Thus, the new investigations promoted by this thesis research intended to consolidate this extension of FL in both senses:

- theoretical foundations of main results in n-DFL,
- main results considering the extended study of n-DI, their main properties and classes contributing to explore other research fields taking advantages of logical structure of n-DFL: (i) formalizing IF-THEN rules of approximate reasoning; (ii) structuring new methods to measure similarity and consensus n-DFS; (iii) extracting parameters as correlation coefficient and entropy supporting the data analysis of applications.

And, see in [Zanotelli et al. 2020] partial relevant thesis results.

4. Conclusion

As a major contribution, the consolidating the research of fuzzy implications on $L_n(U)$ is achieved, considering: (i) analytical studies, defining n-DI, presenting the most desirable properties as neutrality, ordering, (contra-)symmetry, exchange and identity principles, and also discussing their interrelationships and exemplifications; (ii) algebraic aspects mainly related to left- and right-continuity of representable n-dimensional fuzzy t-norms and the generation of n-DI from existing fuzzy implications; (iii) n-dimensional approach of fuzzy implication classes explicitly represented by fuzzy connectives, as (S, N)-implications and QL-implications; (iv) prospective studies of n-DRI, analyzing extended conditions to verify the residuation principle and their characterization based on n-dimensional t-norms and n-DRI; (v) constructive method obtaining n-DRI based on n-dimensional aggregation operators, presenting an exemplification in the solution of a problem involving CIM-MCDM, based on Łukasiewicz n-DRI; and also including (vi) an introductory study considering an n-DI in modeling AR of inference schemes, dealing with the effecting role of such connectives in based-rule fuzzy systems. In addition, taking into account the action of automorphism and fuzzy negations on $L_n(U)$, conjugate and dual operators of n-DI can be respectively obtained, preserving algebraic and analytical properties.

Further work intends to characterize other classes of n-DI, considering their application in the AR models based on inference schemes of deductive systems and construction of admissible linear orders providing new analysis of main properties of fuzzy implications on $L_n(U)$. To sum up, the methodology considered to achieve the main results of this thesis can be extended to multi-dimensional fuzzy interval.

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